

# Zero-modes of Non-Abelian Solitons in Three Dimensional Gauge Theories

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## Abstract

We study non-Abelian solitons of the Bogomol'nyi type in  $\mathcal{N} = 2$  ( $d = 2 + 1$ ) supersymmetric Chern-Simons (CS) and Yang-Mills (YM) theory with a generic gauge group. In CS theory, we find topological, non-topological and semi-local (non-)topological vortices of non-Abelian kinds in unbroken, broken and partially broken vacua. We calculate the number of zero-modes using an index theorem and then we apply the moduli matrix formalism to realize the moduli parameters. For the topological solitons we exhaust all the moduli while we study several examples of the non-topological and semi-local solitons. We find that the zero-modes of the topological solitons are governed by the moduli matrix  $H_0$  only and those of the non-topological solitons are governed by both  $H_0$  and the gauge invariant field  $\Omega$ . We prove local uniqueness of the master equation in the YM case and finally, compare all results between the CS and YM theories.

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# 1 Introduction

Chern-Simons (CS) theory is an alternative gauge theory in odd space-time dimensions (we only consider  $d = 2 + 1$  here), which is very different from Yang-Mills (YM) and Maxwell gauge theories. The CS gauge fields in pure CS theory are non-dynamical, however, they become very interesting upon coupling with other fields. For instance, when considering a CS kinetic term together with a YM kinetic term, the gauge bosons acquire a topological mass in the absence of the Higgs mechanism [1]. The CS gauge theories are interesting both due to their theoretical beauty and also for their experimental applications, such as the fractional quantum Hall effect [2]. The CS term can be induced by quantum radiative corrections, even if the original Lagrangian does not include a bare CS term [3], such as the  $d = 2 + 1$  dimensional quantum electrodynamics (QED).

One of the important features of both CS and YM theories coupled with scalar fields is the existence of solitonic solutions to the classical equations of motion. Thousands of works are made on the study of these solitons. For the CS solitons, early works with/without the Maxwell term were studied in Refs. [4, 5]. In the Abelian CS theory coupled with a complex scalar field, solitons of the self-dual type, viz. solutions satisfying the Bogomol'nyi energy bound and in turn requiring a special sixth-order scalar potential were found in Refs. [6, 7]. The supersymmetric version which corresponds to the self-dual model was obtained in Ref. [8] and the study on zero-modes of the solitons was developed in Ref. [9]. Furthermore, due to the presence of both the CS vacuum and the Higgs vacuum, domain walls and non-topological vortices were found in addition to the familiar topological vortices in the Abelian CS-Higgs (CSH) model [9]. The non-topological soliton is a typical soliton in CS theory which in fact does not exist in the Maxwell-Higgs (or Yang-Mills-Higgs (YMH)) theories. Similar solitons of topological/non-topological kinds were found also in the Maxwell-CS-Higgs (MCSH) theory [10, 11, 12]. Furthermore, a semi-local extension of this type of solitons, viz. in MCSH coupled with two Higgs fields was found both in the topological as well as the non-topological case in Ref. [13]. An extension to the non-Abelian ( $SU(N)$ ) CS theories was also made and non-topological solitons with a global charge were found in Ref. [14]. There are also lots of interesting works on the solitons in the non-relativistic CS theories. Many relevant references on both the relativistic and non-relativistic models can be found in the excellent reviews [15, 16]. The mentioned works on the CS solitons were established

in the early ‘90s and we shall refer to all these as solitons of the Abelian kind.<sup>1</sup>

The second era of the CS solitons recently started when a new type of the topological CS soliton of the non-Abelian (NA) kind<sup>2</sup> was found in the  $U(N)$  CS theory coupled with  $N$  Higgs fields in the fundamental representation [21]. The NA CS vortex found so far is quite similar to the NA YM vortex [17, 18, 19, 20]. It lives in the color-flavor locked, broken vacuum (i.e. the Higgs vacuum) and has internal orientational zero-modes,  $\mathbb{C}P^{N-1}$ . These internal orientational modes are related to the color and flavor degrees of freedom, which make the solitons being truly of the NA kind. Soon after its discovery, the NA semi-local vortices were found in the  $U(N)$  CS theory with  $N_f > N$  fundamental Higgs fields [22, 23]. The dynamics of the NA vortices was studied in Ref. [24] and the NA vortices with a globally conserved charge was found in Ref. [25]. The topological solitons of both local, semi-local and fractional kind in the NA CS theory with an arbitrary gauge group of the form  $U(1)$  times a simple group have also been studied very recently in Refs. [26, 27]. Thus, the study on the CS solitons is being revived now.

The NA CS solitons found so far are all topological solitons in the broken vacuum (the Higgs vacuum). This is because their discovery [21] was inspired by the NA YM vortex which is indeed a topological soliton living in the broken (Higgs) vacuum. However, we know that there are also non-topological solitons in the Abelian CS models [7]. So it is natural to ask ourselves if non-topological or other solitons of the NA kind exist or not, in NA CS theories. With this question in mind, we will be concerned in this paper with vortices of all possible non-Abelian kinds in the NA CS gauge theories with the gauge group  $G$  being general;  $G = U(1) \times G'$  with  $G'$  is an arbitrary simple group. As if it was not broad enough a spectrum, we will make every statement throughout the paper be independent of the gauge group. Beyond our naive expectation, we will find semi-local NA vortices of both topological and non-topological types in the models where the number of flavors is less than or equal to the number of colors.

We will investigate not only single solitons but also multiple solitons, viz. we will focus on the zero-modes of the Bogomol’nyi solutions. To this end, we will count the number of zero-modes by using an index theorem technique. Then we will extend the moduli matrix method for explicit realization of the counted moduli. The moduli matrix formalism was first introduced in the studies of domain wall systems [28, 29] but later developed to encompass vortex systems

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<sup>1</sup>In CS theory as well as YM theory, the vortex in  $SU(N)$  theories is basically of the Abelian kind due to the first homotopy group characterizing the vortices being  $\pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N$  and hence these vortices do not carry non-Abelian orientational moduli, as opposed to those we discuss next.

<sup>2</sup>The non-Abelian vortices were first found in the supersymmetric YM theories in four dimensions [17, 18]. There are many works on the NA vortices in YM theories and we shall refer to the literature for the many results which however are nicely summarized in several review articles [19, 20].

[30], which has proven an invaluable tool in soliton studies [31, 32, 33, 20]. The moduli matrix formalism has already been introduced to the CS models in Refs. [26, 27] for the topological solitons. Here we will try to further develop it to cover all the solitonic solutions in the system, including the non-topological solitons in the unbroken and partially broken vacua. It turns out that the moduli matrix formalism, which has proven itself incredibly useful for topological solitons in YM theories, provides us with a common framework for studying all kinds of solitons not only in YM but also in CS theories and especially in all vacua of the systems at hand. In order to find similarities and differences among the solitons in YM and CS theories, we will show all the results parallelly in the two cases. Most of the statements concerning the YM vortex (up to Sec. 4) are just a review of the known results, however written in a more transparent way with respect to the gauge group. A new result – and one of the main results of this paper – for the YM vortex is on the uniqueness problem of the so-called master equation (Sec. 5).

As we have already mentioned, we will pay special attention to the moduli space of solutions (or simply moduli space). It is one of the most important properties of solitons – of topological or non-topological kind. The moduli space has a certain dimension characterizing the soliton and the gauge theory (or string theory configuration) at hand and some local coordinates on it which can be identified with the bosonic zero-modes of the system describing the soliton. The number of bosonic zero-modes gives the aforementioned (complex) dimension of the moduli space. The moduli space per se is a classical statement which eventually, far in the infrared, will be modified by quantum corrections. However, this information usually is available only through the knowledge about the classical moduli space supplemented by calculations of a corresponding sigma model describing the low-energy effective theory of the soliton under consideration.

A further powerful technique often used in soliton physics is the moduli space approximation or the so-called geodesic approximation due to a seminal paper [34] which was first applied to monopole scattering and thereafter to a vast number of other soliton configurations in the literature.

Index theorems have proven to be very powerful tools in physics. Needless to mention is the Witten's index indicating if a theory at hand can have its supersymmetry broken spontaneously or not [35]. There is also the Hopf index theorem in sigma models giving the number of vacua in a class theories [36]. An index closely related to what we are interested in here, is the Atiyah-Singer [37] index theorem, which has been used to count the physical parameters describing instantons after the configuration space has been compactified [38]. Finally, there is a generalization of this index to non-compact manifolds, namely the Callias index theorem which counts the zero-modes of a Dirac operator minus the zero-modes of the corresponding adjoint operator [39].

This is a very useful technique in soliton physics and especially in BPS systems where the fluctuations of the BPS equations can easily be written as a Dirac operator acting on a (vector) space of fluctuations. To mention a few cases in the literature, this technique has been applied to monopoles [40] and Abelian-Higgs vortices [41]; to Abelian CS vortices [9]; to Maxwell-CS vortices [11]; to non-Abelian  $U(N)$  vortices [17] and to domain walls of Abelian kind [42] and non-Abelian kind [43].

The organization of the paper will be as follows. In Sec. 2 we set up our theories, the YM-Higgs model and the NA CS-Higgs model. Here we will review the basic properties including the BPS-equations of both theories. In Sec. 3, we will derive a generalized formula for the Callias-type index of a certain class of BPS systems including both our models under consideration. In Sec. 4 we will review and develop the moduli matrix formalism to realize the moduli parameters and also explain the new types of solitons, namely the NA non-topological solitons. In Sec. 5 we will confront the long-standing problem of the uniqueness of the master equations. We find a relation between the variation of the master equations and the vanishing theorems studied in Sec. 3. We conclude the paper with a discussion and outlook in Sec. 6. Secs. 3.2 and 4.1 are reviews of known results, so the reader who is familiar with the NA YM vortices can skip them.

## 2 The model and notation

We begin with the  $\mathcal{N} = 2$  supersymmetric Yang-Mills-Chern-Simons (YMCS) theory coupled with Higgs fields in  $d = 2 + 1$  dimensions. In order to make the following arguments applicable to a wide class of gauge theories, we will not specify the gauge group unless we make some explicit examples. Indeed, we will take the gauge group to be on the form  $G = (U(1) \times G') / \mathbb{Z}_{n_0}$ , where  $G'$  is always a simple group. When we will make some examples we will use the gauge group  $G$  with  $G' = SU(N), SO(N), USp(N)$ . In this case we can choose either  $N = 2M$  or  $N = 2M + 1$  for  $SO(N)$  whereas  $N = 2M$  for  $USp(N)$ . Here  $\mathbb{Z}_{n_0}$  is the center of  $G'$ , see Table 1.

$G' \parallel$	$SU(N)$	$SO(2M)$	$USp(2M)$	$SO(2M + 1)$
$n_0 \parallel$	$N$	2	2	1

Table 1: Various values of the greatest common divisor (gcd) of the Abelian charges of all the  $G'$  invariants which have a non-zero VEV at infinity (in the Higgs phase).

We will use the standard convention for the Hermitian generators

$$\text{Tr}(t^\alpha t^\beta) = \frac{1}{2} \delta^{\alpha\beta} , \quad t^0 = \frac{\mathbf{1}_N}{\sqrt{2N}} , \quad (2.1)$$

for  $\alpha, \beta = 0, 1, 2, \dots, \dim(G')$ , while the index of  $G'$  will be denoted by  $a, b = 1, 2, \dots, \dim(G')$ .

The  $G'$  vector multiplet contains the gauge fields and a real adjoint scalar field  $\{A_\mu^a, \phi^a\}$ . The  $U(1)$  vector multiplet also has the corresponding gauge fields as well as a real singlet scalar field  $\{A_\mu^0, \phi^0\}$ . We consider  $N_f$  Higgs fields  $H_r^A$  ( $r = 1, 2, \dots, N_f$ ;  $A = 1, 2, \dots, N_f$ ) in the fundamental representation of  $G'$  with uniform  $U(1)$  charge. In the following we will use a matrix notation where  $H$  is an  $N \times N_f$  dimensional matrix, so that the gauge symmetry acts from the left-hand side and the flavor symmetry acts from the right-hand side. We will use the following compact notation for the construction of  $\mathfrak{g}$  and  $\mathfrak{g}'$  algebra valued fields, respectively, as follows

$$\phi = \sum_{\alpha=0}^{\dim G'} \phi^\alpha t^\alpha , \quad \hat{\phi} = \sum_{a=1}^{\dim G'} \phi^a t^a . \quad (2.2)$$

We are now ready to write down the Lagrangian. Since all the fermions do not play an important role in the following argument for the solitons under consideration, we show only the bosonic part

$$\begin{aligned} \mathcal{L}_{\text{YMCSH}} = & -\frac{1}{4g^2} (F_{\mu\nu}^a)^2 + \frac{1}{2g^2} (\mathcal{D}_\mu \phi^a)^2 - \frac{\mu}{8\pi} \epsilon^{\mu\nu\rho} \left( A_\mu^a \partial_\nu A_\rho^a - \frac{1}{3} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) \\ & - \frac{1}{4e^2} (F_{\mu\nu}^0)^2 + \frac{1}{2e^2} (\partial_\mu \phi^0)^2 - \frac{\kappa}{8\pi} \epsilon^{\mu\nu\rho} A_\mu^0 \partial_\nu A_\rho^0 \\ & + \text{Tr} \left[ \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger \right] - V_{\text{YMCSH}} , \end{aligned} \quad (2.3)$$

$$V_{\text{YMCSH}} = \frac{g^2}{2} \left\{ \text{Tr} \left[ \left( HH^\dagger - \frac{\mu}{2\pi} \phi \right) t^a \right] \right\}^2 + \frac{e^2}{2} \left\{ \text{Tr} \left[ \left( HH^\dagger - \frac{\kappa}{2\pi} \phi - \frac{\xi}{N} \mathbf{1}_N \right) t^0 \right] \right\}^2 . \quad (2.4)$$

Our conventions are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] , \quad \mathcal{D}_\mu H = (\partial_\mu + i A_\mu) H , \quad \mathcal{D}_\mu \phi = \partial_\mu \phi + i [A_\mu, \phi] . \quad (2.5)$$

The first line is the kinetic term of the  $G'$  vector multiplet and the non-Abelian CS term with the gauge coupling constant  $g$  and the CS coupling constant  $\mu$  which has to be an integer in the quantum theory in order to preserve gauge invariance (up to large gauge transformations). The second line is the contribution from the Abelian vector multiplet with  $e$  being the gauge coupling constant and  $\kappa$  being the CS coupling constant which can take on any (real) value. The potential consists of two terms. The first term is the  $D$ -term of  $G'$  vector multiplet while the second term is that of the  $U(1)$  vector multiplet.

Although we have turned off the terms including fermions, one can still smell the supersymmetric nature in the special relation between the gauge and scalar coupling constants. The so-called Fayet-Iliopoulos (FI) parameter  $\xi$  has been chosen to be positive which ensures stable supersymmetric (SUSY) vacua. In order to simplify the notation we define also the parameter

$$v \equiv \sqrt{\frac{\xi}{N}} > 0 . \quad (2.6)$$

Note that the model has  $SU(N_f)$  flavor symmetry.

If we completely discard the Higgs fields  $H$  (i.e. setting  $H \equiv 0$ ) of the full Lagrangian (2.4), the FI term in Eq. (2.4) can be absorbed by a constant shift in  $\phi^0$ . Hence, the vacuum  $\phi = 0$  is in the symmetric phase where no symmetries are broken. Although the Higgs mechanism is not at work, the vector multiplet acquires a topological mass which is given by

$$m_\kappa = \frac{\kappa e^2}{4\pi} , \quad m_\mu = \frac{\mu g^2}{4\pi} . \quad (2.7)$$

Here  $m_\kappa$  is the mass of the Abelian gauge fields  $A_\mu^0$  as well as the real scalar field  $\phi^0$  while  $m_\mu$  is the mass of  $G'$  gauge fields  $A_\mu^a$  as well as the real adjoint fields  $\phi^a$ .

In the following we will study the above described model in two limits, namely  $\kappa, \mu \rightarrow 0$  which reduces the model to the Yang-Mills-Higgs (YMH) theory without the CS interactions and  $e, g \rightarrow \infty$  which reduces it to the “pure” Chern-Simons-Higgs (CSH) theory without dynamical gauge fields. We will leave the intermediate case of both kinetic terms at finite coupling, viz. YMCS-Higgs theory for a companion paper [44].

## 2.1 Yang-Mills-Higgs theory

Taking the limit  $\kappa, \mu \rightarrow 0$  eliminates the CS kinetic term for the gauge fields and decouples the adjoint scalar field from the  $D$ -terms. Hence the bosonic Lagrangian density can now be written as<sup>3</sup>

$$\begin{aligned} \mathcal{L}_{\text{YMH}} = & -\frac{1}{4g^2} (F_{\mu\nu}^a)^2 + \frac{1}{2g^2} (\mathcal{D}_\mu \phi^a)^2 - \frac{1}{4e^2} (F_{\mu\nu}^0)^2 + \frac{1}{2e^2} (\partial_\mu \phi^0)^2 \\ & + \text{Tr} \left[ \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger \right] - \frac{g^2}{2} \left\{ \text{Tr} [H H^\dagger t^a] \right\}^2 - \frac{e^2}{2} \left\{ \text{Tr} [(H H^\dagger - v^2 \mathbf{1}_N) t^0] \right\}^2 . \end{aligned} \quad (2.8)$$

In the case of  $G' = SU(N)$  there is a unique Higgs vacuum where  $G$  is completely broken

$$H = v \mathbf{1}_N , \quad \phi = 0 . \quad (2.9)$$

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<sup>3</sup> This model can be trivially embedded in  $3+1$  dimensions where the vortex solutions describe strings instead of particles in which case the supersymmetry allowed by the given potential is  $\mathcal{N} = 1$  in  $3+1$  dimensions. Either point of view is consistent with our discussion in the following, i.e. formally there will be no difference.



The Higgs phase is in the color-flavor locking phase where the diagonal global symmetry is preserved

$$U(N) \times SU(N_f) \rightarrow SU(N)_{\text{c+f}} . \quad (2.10)$$

The vacuum is gapped and there are no massless modes, while the mass spectrum is given by

$$m_e = ve , \quad m_g = vg . \quad (2.11)$$

Thanks to supersymmetry, the vector multiplets and the chiral multiplets have the same masses, viz. the Abelian gauge fields  $A_\mu^0$ , the real scalar field  $\phi^0$  and the real part of the trace part of  $H$  all have the same mass  $m_e$ . On the other hand, the  $SU(N)$  gauge fields  $A_\mu^a$ , the real adjoint fields  $\phi^a$  and the real part of the traceless part of  $H$  all have the mass  $m_g$ .

For  $G' = SO(N), USp(N)$  the vacuum structure is much more complicated, see Ref. [45]. We will however, only consider the same vacuum (2.9) for the reason that it preserves the maximal global color-flavor locking symmetry

$$G \times SU(N_f) \rightarrow G'_{\text{c+f}} . \quad (2.12)$$

The masses of the vector multiplets remain the same as in the  $G' = SU(N)$  case. Those for the trace part and some of the traceless part of  $H$  are unchanged but the rest of the fields contained in  $H$  become massless (i.e. due to flat directions) since the number of gauge fields is not sufficient for the Higgs mechanism to eat all the massless fields.

Let us consider topological solitons in this model. We are interested in the vortex which is usually called the non-Abelian vortex. It is a natural extension of the Nielsen-Olesen vortex of the Abelian-Higgs model. Performing a Bogomol'nyi trick on the Hamiltonian under the assumption that the configurations are static, i.e. the energy in the  $\mathbb{C}$ -plane

$$T = \int_{\mathbb{C}} \left( \frac{1}{2g^2} (F_{12}^a - g^2 \text{Tr} [HH^\dagger t^a])^2 + \frac{1}{2e^2} (F_{12}^0 - e^2 \text{Tr} [(HH^\dagger - v^2 \mathbf{1}_N) t^0])^2 \right. \\ \left. + \frac{1}{2g^2} (\mathcal{D}_i \phi^a)^2 + \frac{1}{2e^2} (\partial_i \phi^0)^2 + \text{Tr} \left[ 4 |\bar{\mathcal{D}} H|^2 - v^2 F_{12} - i \epsilon^{ij} \partial_i \{ (\mathcal{D}_j H) H^\dagger \} \right] \right) , \quad (2.13)$$

we can read off the BPS tension which is the lower bound

$$T_{\text{BPS}} = -v^2 \int_{\mathbb{C}} \text{Tr} [F_{12}] = 2\pi v^2 N \nu = 2\pi \xi \nu > 0 . \quad (2.14)$$

Here  $\nu$  is the Abelian winding number determined as

$$\nu = -\frac{1}{2\pi N} \int_{\mathbb{C}} \text{Tr} [F_{12}] = \frac{k}{n_0} , \quad k \in \mathbb{Z}_{>0} , \quad (2.15)$$

where  $n_0$  is the center of the gauge group  $\mathbb{Z}_{n_0}$  (see Table 1)<sup>4</sup>. Finally, from Eq. (2.13) we have the BPS equations ( $\phi^\alpha = 0$ )

$$\bar{\mathcal{D}}H = 0, \quad F_{12}^\alpha = g_\alpha^2 \text{Tr} [(HH^\dagger - v^2 \mathbf{1}_N) t^\alpha] , \quad (2.16)$$

where  $g_\alpha$  stands for the gauge coupling constants ( $g_0 = e$  and  $g_a = g$ ) and  $\alpha$  is not summed over. The non-Abelian part of the second equation is expressed in matrix notation as

$$\hat{F}_{12} = g^2 \langle HH^\dagger \rangle_{G'} , \quad (2.17)$$

where we have introduced a bracket for a projection operation

$$\langle X \rangle_{G'} \equiv \text{Tr} [X t^a] t^a , \quad (2.18)$$

for an arbitrary  $N \times N$  matrix  $X$ .

In the special case of  $e = g$  we can simplify the second BPS equations to the following combined equation

$$F_{12} = e^2 \langle HH^\dagger - v^2 \mathbf{1}_N \rangle_G , \quad (2.19)$$

where we have defined

$$\langle X \rangle_G \equiv \text{Tr} [X t^\alpha] t^\alpha = \frac{1}{2N} \text{Tr} [X] \mathbf{1}_N + \langle X \rangle_{G'} . \quad (2.20)$$

Note that  $\langle X \rangle_G = X/2$  holds if  $X \in G$ .

The above special case of  $e = g$  is not only simple but also has the advantage that solutions to the Abelian BPS equation automatically solve the non-Abelian BPS equations. Hence, without solving the non-Abelian equations, we already have a known class of solutions which are essentially Abelian embeddings. All the minimally winding solutions are indeed in this class. Of course, we have to solve the non-Abelian system for the higher winding vortices which are out of the class.

In particular, we can write the explicit form of  $\langle X \rangle_{G'}$  with  $G' = SU(N)$ ,  $SO(N)$  and  $USp(N)$  as

$$\langle X \rangle_{G'} = \begin{cases} \frac{1}{2} (X - \frac{1}{N} \text{Tr}[X] \mathbf{1}_N) , & G' = SU(N) , \\ \frac{1}{4} (X - J^\dagger X^\text{T} J) , & G' = SO(N), USp(N) . \end{cases} \quad (2.21)$$

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<sup>4</sup>  $n_0$  is the greatest common divisor (gcd) of the Abelian charges of all the  $G'$  invariants which have a non-zero VEV at infinity in the Higgs phase, see Ref. [46].

For  $U \in G' = SO(N), USp(N)$  we will use the invariant rank-two tensor  $J$  defined as

$$U^T J U = J, \quad J^\dagger = \epsilon J, \quad J^\dagger J = \mathbf{1}_N, \quad (2.22)$$

with  $\epsilon = +1$  for  $SO(N)$  and  $\epsilon = -1$  for  $USp(N)$ . Throughout the paper we will adapt the basis in which  $J$  is

$$J = \begin{pmatrix} 0 & \mathbf{1}_M \\ \epsilon \mathbf{1}_M & 0 \end{pmatrix}, \quad (2.23)$$

for  $SO(2M)$  ( $\epsilon = +1$ ) and  $USp(2M)$  ( $\epsilon = -1$ ) while for  $SO(2M+1)$  instead

$$J = \begin{pmatrix} 0 & \mathbf{1}_M & 0 \\ \mathbf{1}_M & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.24)$$

Let us give some examples of minimal winding solutions in the case of  $G' = SU(4), SO(4)$  and  $USp(4)$

$$SU(4) : \quad \begin{cases} H = \text{diag} (H_{k=1}^{\text{ANO}}, v, v, v) \\ F_{12} = \text{diag} (F_{12,k=1}^{\text{ANO}}, 0, 0, 0) \end{cases}, \quad (2.25)$$

$$SO(4), USp(4) : \quad \begin{cases} H = \text{diag} (H_{k=1}^{\text{ANO}}, H_{k=1}^{\text{ANO}}, v, v) \\ F_{12} = \text{diag} (F_{12,k=1}^{\text{ANO}}, F_{12,k=1}^{\text{ANO}}, 0, 0) \end{cases}, \quad (2.26)$$

where  $(H_k^{\text{ANO}}, F_{12,k}^{\text{ANO}})$  stands for  $k \in \mathbb{Z}_{\geq 0}$  winding coaxial-vortex solution in the Abelian theory. Generic non-Abelian solutions in the  $SU(4)$  and  $USp(4)$  models can be generated by acting with the global symmetry of the system on the above solutions. There exist further solutions which belong to the different classes in the  $SO(4)$  cases.

The most significant feature of the non-Abelian vortex is the presence of the so-called orientational zero-modes associated with the spontaneous breaking of the color-flavor locked symmetry. In the above example of  $G' = SU(4)$ , one can easily see that  $SU(4)_{\text{c+f}}$  obeyed in the vacuum ( $r \rightarrow \infty$ ), is in fact broken to  $U(3)_{\text{c+f}}$  at the center of the vortex where  $H_{k=1}^{\text{ANO}} \rightarrow 0$ . Thus the Nambu-Goldstone zero-modes of the single non-Abelian vortex in the  $G' = SU(4)$  theory are

$$\mathcal{M}_{SU(4)}^{k=1} = \mathbb{C} \times \mathbb{C}P^3. \quad (2.27)$$

Here the first factor,  $\mathbb{C}$ , corresponds to the position zero-modes. Similarly, for the case of  $G' = USp(4)$ , the vortex breaks  $USp(4)_{\text{c+f}}$  down to  $U(2)_{\text{c+f}}$ . Hence, in this case the zero-modes are

$$\mathcal{M}_{USp(4)}^{k=1} = \mathbb{C} \times \frac{USp(4)}{U(2)}, \quad (2.28)$$

while for  $G' = SO(4)$  there exists a topological charge in addition to the vortex number, which is called the  $\mathbb{Z}_2$  charge [47]. The moduli space is then split into two copies of the space [48, 47] given by the breaking of  $SO(4)_{\text{c+f}}$  down to  $U(2)_{\text{c+f}}$  as

$$\mathcal{M}_{SO(4)}^{k=1} = \mathbb{C} \times \left[ \frac{SO(4)}{U(2)} \Big|_{\mathbb{Z}_2=+} \cup \frac{SO(4)}{U(2)} \Big|_{\mathbb{Z}_2=-} \right]. \quad (2.29)$$

In the case of  $G' = SO(N)$ , with  $N > 4$  the zero-modes are somewhat more elaborated, so we will not explain the details here, which however can be found in Ref. [47].

## 2.2 Chern-Simons-Higgs theory

The second limit under consideration is  $e, g \rightarrow \infty$ , i.e. the strong gauge coupling limit where the kinetic terms of the vector multiplets vanish. Then the original Lagrangian density (2.4) reduces to the CSH model (see Ref. [26])

$$\begin{aligned} \mathcal{L}_{\text{CSH}} = & -\frac{\mu}{8\pi} \epsilon^{\mu\nu\rho} \left( A_\mu^a \partial_\nu A_\rho^a - \frac{1}{3} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) - \frac{\kappa}{8\pi} \epsilon^{\mu\nu\rho} A_\mu^0 \partial_\nu A_\rho^0 \\ & + \text{Tr} \left[ \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger \right] - V_{\text{CSH}}, \end{aligned} \quad (2.30)$$

$$V_{\text{CSH}} = \text{Tr} [\phi^2 H H^\dagger]. \quad (2.31)$$

The sixth-order scalar potential  $V_{\text{CSH}}$  is given by eliminating the adjoint scalar fields  $\phi^\alpha$  from Eq. (2.4) by

$$\phi^\alpha = \frac{4\pi}{\kappa_\alpha} \text{Tr} \left[ (H H^\dagger - v^2 \mathbf{1}_N) t^\alpha \right], \quad \alpha = 0, 1, 2, \dots, \dim(G'), \quad (2.32)$$

with  $\kappa_0 = \kappa$  and  $\kappa_a = \mu$ , for  $a = 1, 2, \dots, \dim(G')$ , and  $\alpha$  is not summed over on the right-hand side.

There exist three types of vacua in this model:

- Symmetric phase: One vacuum is in the completely symmetric phase (the CS phase) where  $\langle H \rangle = 0$  and no symmetries are broken. The vector multiplets are decoupled, so the Higgs fields are the only dynamical degrees of freedom with the mass

$$m_H = |\phi^0| = \frac{4\pi v^2}{\kappa} \sqrt{\frac{N}{2}}. \quad (2.33)$$

- Asymmetric phase: There is also a vacuum in the Higgs phase  $\langle H \rangle = v \mathbf{1}_N$ , where the gauge symmetry is completely broken. The mass of Higgs fields is the same as that of gauge fields

due to supersymmetry. The Abelian gauge fields  $A_\mu^0$  and the real part of the trace part of  $H$  have the same mass  $m_{\kappa\infty}$ , while the  $G'$  gauge fields  $A_\mu^a$ , the real part of traceless part of  $H$  have the mass  $m_{\mu\infty}$ .

$$m_{\kappa\infty} = \frac{m_e^2}{m_\kappa} = \frac{4\pi v^2}{\kappa} , \quad m_{\mu\infty} = \frac{m_g^2}{m_\mu} = \frac{4\pi v^2}{\mu} . \quad (2.34)$$

The Higgs vacuum  $\langle H \rangle = v \mathbf{1}_N$  is unique if  $G' = SU(N)$  but there exist flat directions in the cases of  $G' = SO(N), USp(N)$  as we have seen in the previous section.

- Intermediate phases: In between there is a variety of partially broken phases.

In the model with  $G' = SU(N)$  and  $N_f = N$ , the vacua are labeled by an integer  $m = 0, 1, 2, \dots, N$  as follows

$$H^{(m)} = \text{diag}(\underbrace{v, v, \dots, v}_m, \underbrace{0, \dots, 0}_{N-m}) . \quad (2.35)$$

The vacuum  $m = N$  is the full Higgs vacuum where the symmetry  $U(N)_c \times SU(N)_f$  is broken to  $SU(N)_{c+f}$ . The vacuum  $m = 0$  is the unbroken phase where no symmetries are broken, while in the case of  $1 \leq m \leq N - 1$  we have the intermediate vacua. In this case the symmetry is broken down to  $U(N - m)_c \times S[U(m)_{c+f} \times U(N - m)_f]$ . Note that the vacua with  $m = 0, N$  are unique but the rest are not. The vacuum manifold is a complex Grassmannian manifold given by  $SU(N)_f / S[U(m)_{c+f} \times U(N - m)_f] \simeq Gr_{N,m}$ .

The variety of the vacua results in a variety of vortices which can be either topological or non-topological. In order to derive the BPS equations, we again perform a Bogomol'nyi trick on the tension of the vortex

$$T = \int_{\mathbb{C}} \text{Tr} \left[ |(\mathcal{D}_0 - i\phi) H|^2 + 4 |\bar{\mathcal{D}} H|^2 - v^2 F_{12} - i\epsilon^{ij} \partial_i \{ (\mathcal{D}_j H) H^\dagger \} \right] , \quad (2.36)$$

where  $\phi$  is given by Eq. (2.32). The tension for the BPS saturated vortices is given by

$$T_{\text{BPS}} = -v^2 \int_{\mathbb{C}} \text{Tr}[F_{12}] = 2\pi N v^2 \nu > 0 . \quad (2.37)$$

where  $\nu$  is the  $U(1)$  winding number

$$\nu = -\frac{1}{2\pi\sqrt{2N}} \int_{\mathbb{C}} F_{12}^0 = \begin{cases} \frac{k}{n_0} & \text{for topological solitons} \\ \frac{k+\alpha}{n_0} & \text{for non-topological solitons} \\ \frac{k}{n_0} + \frac{k'+\alpha}{n_0} & \text{for topological and non-topological solitons} \end{cases} , \quad (2.38)$$

where  $k, k'$  are integers and  $\alpha$  is a real number. In the case of Abelian non-topological solitons, the integer part  $k$  is limited as  $\alpha > k + 2$ , see Ref. [49]. For non-Abelian cases, no conditions

for  $\alpha$  and  $k$  are in general known yet. But only in the special case of  $U(N)$  gauge group where the coupling constants are equal  $\kappa = \mu$  and for the diagonal solutions, we can easily extend the Abelian results as  $\alpha_i > k_i + 2$ . Here  $\alpha_i$  ( $\alpha = \sum_i^N \alpha_i$ ) and  $k_i$  ( $k = \sum_i^N k_i$ ) are associated with the non-topological and topological flux of the  $i$ -th  $U(1)$  subgroup of  $U(1)^N \subset U(N)$ .

Let us furthermore define the magnetic flux as follows

$$\Phi \equiv 2\pi N\nu . \quad (2.39)$$

The BPS equations in this case read

$$\bar{\mathcal{D}}H = 0 , \quad \mathcal{D}_0 H = i\phi H , \quad (2.40)$$

which however due to the presence of electric charge density in the vortex have to be accompanied by the Gauss law ( $\alpha$  not summed over)

$$F_{12}^\alpha = -\frac{i4\pi}{\kappa_\alpha} \text{Tr} \left[ \left( (\mathcal{D}_0 H) H^\dagger - H (\mathcal{D}_0 H)^\dagger \right) t^\alpha \right] . \quad (2.41)$$

Combining the equations (2.40) and (2.41) we obtain the following system

$$\bar{\mathcal{D}}H = 0 , \quad F_{12}^\alpha = \frac{4\pi}{\kappa_\alpha} \text{Tr} \left[ \{ \phi, HH^\dagger \} t^\alpha \right] . \quad (2.42)$$

This system is valid for any simple group  $G'$ . In matrix notation, we can explicitly write down the second equation as

$$F_{12}^0 t^0 = \frac{8\pi^2}{N^2 \kappa^2} \text{Tr} [HH^\dagger - v^2 \mathbf{1}_N] \text{Tr} [HH^\dagger] \mathbf{1}_N + \frac{16\pi^2}{N\kappa\mu} \text{Tr} [\langle HH^\dagger \rangle_{G'} HH^\dagger] \mathbf{1}_N , \quad (2.43)$$

$$\hat{F}_{12} = \frac{16\pi^2}{N\mu\kappa} \text{Tr} [HH^\dagger - v^2 \mathbf{1}_N] \langle HH^\dagger \rangle_{G'} + \frac{32\pi^2}{\mu^2} \langle \langle HH^\dagger \rangle_{G'} HH^\dagger \rangle_{G'} , \quad (2.44)$$

where we have used the following identity

$$\begin{aligned} \langle X \langle X \rangle \rangle_{G'} &= \text{Tr} [X t^b t^a] \text{Tr} [X t^b] t^a \\ &= \text{Tr} [X t^a t^b] \text{Tr} [X t^b] t^a + i f^{bac} \text{Tr} [X t^c] \text{Tr} [X t^b] t^a \\ &= \langle \langle X \rangle X \rangle_{G'} , \end{aligned} \quad (2.45)$$

due to the fact that  $f^{bac} = -f^{cab}$  in the basis (2.1). In the equal coupling case  $\kappa = \mu$  we can simplify the BPS equations (2.43) and (2.44) to the following combined equation

$$F_{12} = \frac{32\pi^2}{\kappa^2} \langle \langle HH^\dagger - v^2 \mathbf{1}_N \rangle_G HH^\dagger \rangle_G . \quad (2.46)$$

Now we have seen the differences between the two different BPS systems in Sec. 2.1 and Sec. 2.2. The equation for the Higgs fields  $H$  (Eqs. (2.16) and (2.42)), is common. The difference resides only in the flux equations (2.19) and (2.46), which will eventually become the master equations after applying the moduli matrix method in Sec. 4. We want to study the zero-modes of the BPS solutions to these equations in the subsequent sections.

### 2.2.1 The Abelian Chern-Simons solitons

Let us recall the solitons in the minimal model by choosing  $N = 1$  and  $N_f = 1$ , namely the Abelian CSH model [9]. The BPS equations are

$$\bar{\mathcal{D}}H = 0 , \quad \frac{1}{\sqrt{2}}F_{12}^0 = \frac{8\pi^2}{\kappa^2} (|H|^2 - v^2) |H|^2 . \quad (2.47)$$

It is known that there are two kinds of solitons: i) the topological solitons and ii) the non-topological solitons. Let  $(H^{\text{TP}}, F_{12}^{\text{TP}})$  be a topological solution and  $(H^{\text{NTP}}, F_{12}^{\text{NTP}})$  be a non-topological solution (several numerical solutions have been obtained in Ref. [9]). The topological solitons live in the Higgs vacuum while the non-topological solitons live in the unbroken (CS) vacuum. Therefore, the asymptotic behavior of the Higgs field is

$$|H^{\text{TP}}| \rightarrow v , \quad |H^{\text{NTP}}| \rightarrow 0 , \quad \text{as } r \rightarrow \infty . \quad (2.48)$$

For topological reasons  $H^{\text{TP}}$  must vanish at the center of the vortex but this is not the case for the non-topological soliton

$$H^{\text{TP}} \rightarrow 0 , \quad H^{\text{NTP}} \rightarrow v' , \quad \text{as } r \rightarrow 0 , \quad (2.49)$$

where  $v' \in (0, v)$  is a constant. The topological soliton has a quantized magnetic flux (2.39) whereas the flux of the non-topological one is a continuum

$$\Phi^{\text{TP}} = 2\pi k , \quad \Phi^{\text{NTP}} = 2\pi(k + \alpha) , \quad (2.50)$$

with  $k \in \mathbb{Z}_{\geq 0}$  and  $\alpha > k + 2$ . For both topological and non-topological vortices, the integer  $k$  corresponds to the number of the zeros of the Higgs field  $H$ .

It turns out that the dimension of the moduli space of the Abelian CS solitons is

$$\dim_{\mathbb{C}} \mathcal{M}_{\text{ACS}} = k + \hat{\alpha} - 1 . \quad (2.51)$$

Here  $\hat{\alpha}$  the integer-part of the real number  $\alpha$  [9].

### 2.2.2 The $U(2)$ Chern-Simons solitons

To describe some of the characteristic features of the non-Abelian extension of the CS solitons, we will discuss the simple example of  $G' = SU(2)$  with  $N_f = 2$  in this subsection. We also set  $\kappa = \mu$  so that the Abelian solutions automatically solve the non-Abelian equations

$$\bar{\mathcal{D}}H = 0 , \quad F_{12} = \frac{8\pi^2}{\kappa^2} (HH^\dagger - v^2 \mathbf{1}_2) HH^\dagger . \quad (2.52)$$

Let us first figure out the vacuum structure of the model. There exist three different vacua: (2) the Higgs vacuum, (0) the unbroken (CS) vacuum and (1) intermediate vacua

$$H^{(2)} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \quad H^{(1)} = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}, \quad H^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.53)$$

The symmetry of the vacua is  $SU(2)_{c+f}$  in the Higgs vacuum,  $U(1)_c \times U(1)_{c+f}$  in the intermediate vacuum and finally  $U(2)_c \times SU(2)_f$  in the unbroken vacuum. As already mentioned, the Higgs and unbroken vacua are unique but the intermediate vacuum possesses a continuum. By acting with the  $SU(2)_f$  flavor symmetry, one can transform the above given configuration into a generic point

$$H^{(1)} = v \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & 0 \end{pmatrix}, \quad |\phi_1|^2 + |\phi_2|^2 = 1. \quad (2.54)$$

Since the overall phase of  $\phi_1, \phi_2$  is gauged, the vacuum manifold is  $SU(2)/U(1) \simeq \mathbb{C}P^1$ . One can exchange the first and second rows in the matrix, which however is  $SU(2)_c$  gauge equivalent to the given form.

In the Higgs vacuum we can construct the topological vortex as follows

$$H = \begin{pmatrix} H^{\text{TP}} & 0 \\ 0 & v \end{pmatrix} \rightarrow \begin{cases} \text{diag}(0, v), & \text{as } r \rightarrow 0, \\ \text{diag}(v, v), & \text{as } r \rightarrow \infty. \end{cases} \quad (2.55)$$

Clearly, the color-flavor symmetry  $SU(2)_{c+f}$  respected by the vacuum is spontaneously broken to  $U(1)_{c+f}$  at the center of the vortex. This yields Nambu-Goldstone zero-modes which are the so-called orientational zero-modes in addition to the position (translational) zero-modes. Thus the minimal topological vortex carries the zero-modes [21]

$$\mathcal{M}_{k=1}^{\text{TP}} = \mathbb{C} \times \mathbb{C}P^1. \quad (2.56)$$

In the unbroken phase, we can construct the non-topological solitons as follows

$$H = \begin{pmatrix} H^{\text{NTP}} & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{cases} \text{diag}(v', 0), & \text{as } r \rightarrow 0, \\ \text{diag}(0, 0), & \text{as } r \rightarrow \infty. \end{cases} \quad (2.57)$$

The symmetry of the vacuum  $U(2)_c \times SU(2)_f$  is spontaneously broken to  $U(1)_c \times U(1)_{c+f}$  at the center of the soliton. The spontaneous breaking of the global symmetry leads to the Nambu-Goldstone modes  $SU(2)_f/U(1)_{c+f} \simeq \mathbb{C}P^1$ . For simplicity, let us choose the minimal non-topological soliton (corresponding to  $\hat{\alpha} = 2$ ) which in the Abelian case has only translational



zero-modes. Clearly, with the above symmetry argument it indeed carries the same zero-modes as the topological soliton

$$\mathcal{M}_{\hat{a}=2}^{\text{NTP}} = \mathbb{C} \times \mathbb{C}P^1 , \quad (2.58)$$

where the first  $\mathbb{C}$  corresponds to the translation zero-mode. We will denote this the non-Abelian non-topological soliton.

The intermediate vacua are interesting. We can put either topological or non-topological solitons. Let us first construct the topological soliton. A naive way is simply to embed the Abelian topological solution as

$$H = \begin{pmatrix} H^{\text{TP}} & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{cases} \text{diag}(0, 0) , & \text{as } r \rightarrow 0 , \\ \text{diag}(v, 0) , & \text{as } r \rightarrow \infty . \end{cases} \quad (2.59)$$

Since the symmetry at the vortex center is larger than that of the vacuum, there are no orientational zero-modes. However, this is not the most general solution. The generic solutions can be obtained by embedding not the Abelian solution with  $N_f = 1$  but the Abelian solutions with  $N_f = 2$ , hence the semi-local Abelian vortex solution which have been studied in Ref. [13]. Let  $(H_1^{\text{STP}}, H_2^{\text{STP}})$  be the fields of the semi-local vortex solution in the Abelian theory. Then the generic solutions of the  $U(2)$  model can be written as follows

$$H = \begin{pmatrix} H_1^{\text{STP}} & H_2^{\text{STP}} \\ 0 & 0 \end{pmatrix} \rightarrow \begin{cases} \begin{pmatrix} 0 & v' \\ 0 & 0 \end{pmatrix} , & \text{as } r \rightarrow 0 \\ \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} , & \text{as } r \rightarrow \infty \end{cases} , \quad (2.60)$$

with  $v'$  being a constant which should be determined by the equations of motion. It is known that the semi-local vortex  $(H_1^{\text{STP}}, H_2^{\text{STP}})$  has a complex free parameter, say  $a \in \mathbb{C}$ . We can identify  $|a|$  with the transverse size of the vortex while the phase is the relative  $U(1)$  phase between the first and second Higgs elements. When we send  $a$  to zero, the generic solution becomes the naive embedding solution (2.59). Thus we conclude that the moduli space of the topological solitons in the intermediate vacua is

$$\mathcal{M}_{k=1}^{\text{STP}} = \mathbb{C} \times \mathbb{C} . \quad (2.61)$$

The first  $\mathbb{C}$  is the position and the second corresponds to the semi-local size modulus  $a$ . This embedding solution is easily extended to the  $U(N)$  case. As explained before, there are  $N - 1$  different kinds of intermediate vacua, see Eq. (2.35). Obviously, in the vacua with  $m = 1$  and  $N - m = 1$  we can embed the Abelian semi-local vortex of the topological and non-topological

kind, respectively. In the rest vacua with  $m > 1$  and  $N - m > 1$  we can have non-Abelian semi-local vortices. So far, the semi-local vortices have been observed only in theories in which the number of flavors is greater than that of colors ( $N_f > N$ ), in the case of  $G' = SU(N)$ , see Refs. [22, 23]. Here we first show that the semi-local vortices also exist in the model with  $N_f = N$ . Furthermore, if one likes, one can also consider systems with  $N_f < N$ , for instance  $N_f = 2$  and  $N = 3$ , and construct the semi-local vortex there by choosing an appropriate vacuum.<sup>5</sup>

There exists also a semi-local non-topological soliton in the Abelian theory with  $N_f = 2$ , see Ref. [13]. Let  $(H_1^{\text{SNTP}}, H_2^{\text{SNTP}})$  be a minimal solution (which corresponds to  $\hat{\alpha} = 2$ ). Then we can embed this into the non-Abelian model as follows

$$H = \begin{pmatrix} v & 0 \\ H_1^{\text{SNTP}} & H_2^{\text{SNTP}} \end{pmatrix} \rightarrow \begin{cases} \begin{pmatrix} v & 0 \\ v' & v'' \end{pmatrix} & \text{as } r \rightarrow 0, \\ \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} & \text{as } r \rightarrow \infty. \end{cases} \quad (2.62)$$

The solution has also one complex free parameter so the moduli space is again

$$\mathcal{M}_{\hat{\alpha}=2}^{\text{SNTP}} = \mathbb{C} \times \mathbb{C}. \quad (2.63)$$

In Sec. 4, we will study generic solutions and explicitly see the structure of moduli space by using the moduli matrix method.

## 3 The index theorem

### 3.1 Calculation of the index

Let us first make a general calculation of the index. Later we will apply the results to the YMH and the CSH models considered above.

Suppose the zero-modes  $\eta$  of a set of BPS equations are described by a Dirac-type equation

$$\mathbb{D}\eta = 0, \quad \eta = \begin{pmatrix} \eta_f \\ \eta_a \end{pmatrix}. \quad (3.1)$$

---

<sup>5</sup>In  $G' = SO, USp$  theories it is however different due to the already mentioned complicated vacuum structure, and the existence of the flat directions allows for semi-local vortices even for the minimal number of flavors in order to put the theory on the Higgs branch [45, 46, 27].

where  $\eta$  takes a value in a vector-space generally described by matrices with  $\eta_f$  being an  $N_f \times N$  matrix while  $\eta_a$  is an  $N \times N$  matrix. The inner product on this vector space is defined as

$$(\iota, \eta) \equiv \int_C \text{Tr} \left[ \iota_f^\dagger \eta_f + \iota_a^\dagger \eta_a \right] , \quad (3.2)$$

while the adjoint operator of  $\mathbb{D}$  is denoted by  $\mathbb{D}^\dagger$  and is defined as usual  $(\iota, \mathbb{D}\eta) = (\mathbb{D}^\dagger \iota, \eta)$ .

In this section, we will calculate the index of the operator  $\mathbb{D}$  of the Callias type [39] (i.e. the generalization of the Atiyah-Singer theorem to non-compact manifolds), which is formally

$$\mathcal{I} = \dim(\ker \mathbb{D}) - \dim(\ker \mathbb{D}^\dagger) = \dim(\ker \mathbb{D}^\dagger \mathbb{D}) - \dim(\ker \mathbb{D} \mathbb{D}^\dagger) , \quad (3.3)$$

where we have used that the kernel of  $\mathbb{D}$  ( $\mathbb{D}^\dagger$ ) is the same as  $\mathbb{D}^\dagger \mathbb{D}$  ( $\mathbb{D} \mathbb{D}^\dagger$ ). We will however calculate a slightly different index defined by<sup>6</sup>

$$\hat{\mathcal{I}}(M^2) = \text{tr} \left( \frac{M^2}{\mathbb{D}^\dagger \mathbb{D} + M^2} \right) - \text{tr} \left( \frac{M^2}{\mathbb{D} \mathbb{D}^\dagger + M^2} \right) , \quad (3.4)$$

which clearly gives back the index  $\mathcal{I}$  in the limit  $M^2 \rightarrow 0$  if the system possesses a mass gap, but in most cases it will turn out to be independent<sup>7</sup> of  $M^2$ , hence it will prove useful to evaluate the index in the limit  $M^2 \rightarrow \infty$ . Subtleties arise in the case where the continuum part of the spectrum extends down to zero and one has to be more careful [50]. This is because an unbounded resonance gives a certain contribution to the index. In such cases, we have to subtract the continuum part from  $\hat{\mathcal{I}}$  [9].

Let us now make some general statements for the form of the operators  $\mathbb{D}, \mathbb{D}^\dagger$ . First of all, we decompose the Dirac operators as

$$\mathbb{D} = iD + K , \quad \mathbb{D}^\dagger = iD^\dagger + K^\dagger , \quad (3.5)$$

where  $D$  includes only the differential operators  $\partial, \bar{\partial}$  and  $K, K^\dagger$  contain the remaining parts. This yields

$$\mathbb{D}^\dagger \mathbb{D} = \Delta - L_1, \quad \mathbb{D} \mathbb{D}^\dagger = \Delta - L_2, \quad (3.6)$$

where we have defined

$$\Delta \equiv -D^\dagger D = -DD^\dagger , \quad (3.7)$$

$$L_1 \equiv -iD^\dagger K - iK^\dagger D - K^\dagger K , \quad (3.8)$$

$$L_2 \equiv -iDK^\dagger - iKD^\dagger - KK^\dagger . \quad (3.9)$$

---

<sup>6</sup>We will adopt the notation where  $\text{tr}$  denotes trace over states as well as over the matrices while  $\text{Tr}$  denotes only a matrix trace.

<sup>7</sup>This can be proven for a compact manifold and the result proves to be the same if the fields are sufficiently well-behaved at spatial infinity.

We assume that  $\Delta$  is diagonal and composed by Laplacians, while  $L_{1,2}$  contain only linear differential operators. Expanding in terms of the large mass-squared parameter, we obtain

$$\begin{aligned}\frac{M^2}{\mathbb{D}^\dagger \mathbb{D} + M^2} &= M^2 [P^{-1} + P^{-1} L_1 P^{-1} + P^{-1} L_1 P^{-1} L_1 P^{-1} + \dots] , \\ \frac{M^2}{\mathbb{D} \mathbb{D}^\dagger + M^2} &= M^2 [P^{-1} + P^{-1} L_2 P^{-1} + P^{-1} L_2 P^{-1} L_2 P^{-1} + \dots] ,\end{aligned}\quad (3.10)$$

where we have defined  $P \equiv \Delta + M^2$ . All terms beyond the second order in  $L_i$  will vanish in the large mass limit  $M^2 \rightarrow \infty$ .

Now we will pursue a technique used in Abelian systems [11] to simplify the calculation, which will demonstrate that the terms  $K^\dagger K$  and  $K K^\dagger$  do not contribute to the index. We can express the index in the following four terms as

$$\hat{\mathcal{I}} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 , \quad (3.11)$$

with

$$\mathcal{I}_1 \equiv \lim_{M^2 \rightarrow \infty} i M^2 \text{tr} [P^{-1} (DK^\dagger - K^\dagger D - D^\dagger K + K D^\dagger) P^{-1}] , \quad (3.12)$$

$$\mathcal{I}_2 \equiv \lim_{M^2 \rightarrow \infty} M^2 \text{tr} [P^{-1} (K K^\dagger - K^\dagger K) P^{-1}] , \quad (3.13)$$

$$\mathcal{I}_3 \equiv - \lim_{M^2 \rightarrow \infty} M^2 \text{tr} [P^{-1} (D^\dagger K + K^\dagger D) P^{-1} (D^\dagger K + K^\dagger D) P^{-1}] , \quad (3.14)$$

$$\mathcal{I}_4 \equiv \lim_{M^2 \rightarrow \infty} M^2 \text{tr} [P^{-1} (DK^\dagger + K D^\dagger) P^{-1} (DK^\dagger + K D^\dagger) P^{-1}] . \quad (3.15)$$

The trick now is that  $\mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 = 0$  which can be easily proven by making use of

$$[P^{-1}, D] = [P^{-1}, D^\dagger] = 0, \quad DP^{-1}D^\dagger = D^\dagger P^{-1}D = -\mathbf{1} + M^2 P^{-1}. \quad (3.16)$$

Hence, we can write the index simply as

$$\hat{\mathcal{I}} = \mathcal{I}_1. \quad (3.17)$$

This formula is considerably simpler than the starting point (3.11).

Let us calculate the index for a class of operators which includes our models

$$\mathbb{D} = \begin{pmatrix} i\bar{\mathcal{D}}_{\text{fund}}^T & K_1 \\ K_2 & i\mathcal{D}_{\text{adj}}^T \end{pmatrix}, \quad \bar{\mathcal{D}}_{\text{fund}}^T = \bar{\partial} + i \circ \bar{A}^T, \quad \mathcal{D}_{\text{adj}}^T = \partial - i [A^T, \circ] . \quad (3.18)$$

where the gauge fields  $A, \bar{A}$  of the gauge group  $G$  is a given background configuration. The Dirac operator acts on fields of the following form

$$\eta = \begin{pmatrix} \delta H^T \\ \delta \bar{A}^T \end{pmatrix}, \quad (3.19)$$

where  $\delta H$  is an  $N \times N_f$  matrix-valued field in the fundamental representation of  $G$  whereas  $\delta \bar{A}$  is an  $N \times N$  matrix-valued field in the adjoint representation of  $G$ .  $K_1$  (i.e. an  $N_f \times N$  matrix) and  $K_2$  (i.e. an  $N \times N_f$  matrix) need not be fixed yet, so we leave them as variables until we will need the specific forms. It follows that with  $\mathbb{D} = iD + K$

$$D = \begin{pmatrix} \mathbf{1}_{N_f} \bar{\partial} & 0 \\ 0 & \mathbf{1}_N \partial \end{pmatrix}, \quad K = \begin{pmatrix} -\circ \bar{A}^T & K_1 \\ K_2 & [A^T, \circ] \end{pmatrix}, \quad (3.20)$$

and its adjoint operator  $\mathbb{D}^\dagger = iD^\dagger + K^\dagger$  is defined by

$$D^\dagger = \begin{pmatrix} \mathbf{1}_{N_f} \partial & 0 \\ 0 & \mathbf{1}_N \bar{\partial} \end{pmatrix}, \quad K^\dagger = \begin{pmatrix} -\circ A^T & K_2^\dagger \\ K_1^\dagger & [\bar{A}^T, \circ] \end{pmatrix}, \quad (3.21)$$

where  $K_{1,2}^\dagger$  are the adjoint operators of  $K_{1,2}$ . The Laplacian operator reads

$$\Delta = -D^\dagger D = -DD^\dagger = -\bar{\partial}\partial. \quad (3.22)$$

Now by a simple calculation we can show that

$$DK^\dagger - K^\dagger D - D^\dagger K + KD^\dagger = \begin{pmatrix} \circ(\partial \bar{A}^T - \bar{\partial} A^T) & * \\ * & [\partial \bar{A}^T - \bar{\partial} A^T, \circ] \end{pmatrix}, \quad (3.23)$$

independently of  $K_{1,2}$ .<sup>8</sup>

We can now easily calculate the index by means of Eq. (3.17)

$$\begin{aligned} \lim_{M^2 \rightarrow \infty} \hat{\mathcal{I}} &= iM^2 \int d^2x \operatorname{tr} \langle x | P^{-1} (DK^\dagger - K^\dagger D - D^\dagger K + KD^\dagger) P^{-1} | x \rangle \\ &= - \sum_{N_f} \int_{\mathbb{C}} i \operatorname{Tr} (\bar{\partial} A - \partial \bar{A}) \int \frac{d^2k}{(2\pi)^2} \frac{M^2}{(\frac{1}{4}k^2 + M^2)^2} \\ &= N_f N \nu, \end{aligned} \quad (3.24)$$

which turns out to be independent of  $M^2$  and the Abelian winding number  $\nu$  is defined in Eqs. (2.15) and (2.38), for YMH and CSH theories, respectively.

The result (3.24) is quite general and can be applied to all the solitons under consideration in this paper. Namely, the index (3.24) is applicable to all the models considered in this paper: YMH and CSH theories with a general gauge group

$$G = [U(1) \times G'] / \mathbb{Z}_{n_0}.$$

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<sup>8</sup>Even though the index theorem calculation does not depend on the constants  $K_{1,2}$ , they are needed to derive the vanishing theorem, see below.

Next we want to show that the kernel of  $\mathbb{D}^\dagger$  is indeed vanishing. This statement is usually denoted the vanishing theorem, and we will need to compute those case by case. The applicability of the vanishing theorem makes the index equal to the number of zero-modes of the system described by the operator  $\mathbb{D}$ .

## 3.2 Yang-Mills-Higgs theory

### 3.2.1 The index for Yang-Mills vortices

The index theorem in the case of  $G' = SU(N)$  YM theory was already made in Ref. [17] while it was made with a general group in Ref. [47], hence we will just briefly review the calculation here as we will need it later.

In what follows, we set  $e = g$ , in order to make the expressions more compact. Our starting point is the BPS equations (2.16) and (2.19). Let  $\{H, H^\dagger, A, \bar{A}\}$  be a BPS solution. Then we consider small fluctuations around it as  $H \rightarrow H + \delta H$ ,  $H^\dagger \rightarrow H^\dagger + \delta H^\dagger$ ,  $A \rightarrow A + \delta A$  and  $\bar{A} \rightarrow \bar{A} + \delta \bar{A}$ . The fluctuations obey the following linearized BPS equations

$$i\bar{\mathcal{D}}_f \delta H - \delta \bar{A} H = 0 , \quad (3.25)$$

$$i\bar{\mathcal{D}}_a \delta A - i\mathcal{D}_a \delta \bar{A} = \frac{e^2}{2} (\langle \delta H H^\dagger \rangle_G + \langle H \delta H^\dagger \rangle_G) , \quad (3.26)$$

where we have used that  $\delta F_{12} = 2i (\bar{\mathcal{D}}_a \delta A - \mathcal{D}_a \delta \bar{A})$ . We need to introduce a gauge fixing in order not to count non-physical degrees of freedom and it will prove convenient to choose [17]

$$i\bar{\mathcal{D}}_a \delta A + i\mathcal{D}_a \delta \bar{A} = \frac{e^2}{2} (-\langle \delta H H^\dagger \rangle_G + \langle H \delta H^\dagger \rangle_G) . \quad (3.27)$$

Combining Eqs. (3.26) and (3.27), we can now write these equations in the form of Eq. (3.1) with the following Dirac operator

$$\mathbb{D} = \begin{pmatrix} i\bar{\mathcal{D}}_f^T & -H^T \\ \frac{e^2}{2} \langle \bar{H} \circ \rangle_G & i\mathcal{D}_a^T \end{pmatrix} , \quad \mathbb{D}^\dagger = \begin{pmatrix} i\mathcal{D}_f^T & \frac{e^2}{4} H^T \\ -2 \langle \bar{H} \circ \rangle_G & i\bar{\mathcal{D}}_a^T \end{pmatrix} , \quad \eta = \begin{pmatrix} \delta H^T \\ \delta \bar{A}^T \end{pmatrix} , \quad (3.28)$$

where  $\bar{H}$  is just the complex conjugate of  $H$  while  $H^\dagger$  is the Hermitian conjugate as usual. Since this operator is in the class of operators (3.18), we can immediately apply the result of the index (3.24). From Eqs. (2.15) and (3.24), we find the index

$$\mathcal{I} = \lim_{M^2 \rightarrow \infty} \hat{\mathcal{I}} = \frac{N_f N k}{n_0} . \quad (3.29)$$

### 3.2.2 Vanishing theorem for Yang-Mills vortices

In order to establish that the index does in fact correspond to the number of bosonic zero-modes of the BPS vortex solutions, we need to demonstrate that the dimension of the kernel of the adjoint operator is indeed zero

$$\dim(\ker \mathbb{D}^\dagger) = 0 . \quad (3.30)$$

The question is if there exist normalizable zero-modes  $\chi$  for  $\mathbb{D}^\dagger$

$$\mathbb{D}^\dagger \chi = 0 , \quad \chi = \begin{pmatrix} \chi_f^T \\ \chi_a^T \end{pmatrix} , \quad (3.31)$$

where  $\chi_f^T$  is an  $N_f$ -by- $N$  matrix and  $\chi_a^T$  is an  $N$ -by- $N$  matrix and they obey the following equations

$$\mathcal{D}_f \chi_f = \frac{ie^2}{4} \chi_a H , \quad \bar{\mathcal{D}}_a \chi_a = -i2 \langle \chi_f H^\dagger \rangle_G , \quad (3.32)$$

which can be combined into the following equation

$$\mathcal{D}_a \bar{\mathcal{D}}_a \chi_a = \frac{e^2}{2} \langle \chi_a H H^\dagger \rangle_G . \quad (3.33)$$

In the Abelian case this is simply the Schrödinger-type equation

$$\left( -\partial \bar{\partial} + \frac{e^2}{4} |H|^2 \right) \chi_a = 0 . \quad (3.34)$$

Since the Schrödinger potential is positive semi-definite, we can immediately conclude that  $\chi_a = 0$  and this in turn tells us from the second equation of (3.32) that also  $\chi_f$  needs to vanish on the entire plane. It follows that there are no normalizable massless modes in the Abelian system.

For the non-Abelian case, the Schrödinger equation seems to be somewhat more complicated. There is however a nice trick to solve this problem, which is to write the square of the adjoint operator on a given state as a sum of squared terms. It was first introduced in Ref. [17] in the case of  $G' = SU(N)$ , and was also applied to a general group in Ref. [47]. The starting point is to take the complex norm of  $\mathbb{D}^\dagger \chi = 0$  as

$$\begin{aligned} 0 &= (\mathbb{D}^\dagger \chi, \mathbb{D}^\dagger \chi) \\ &= \int_{\mathbb{C}} \text{Tr} \left[ |i\mathcal{D}_f \chi_f + \chi_a H|^2 + |i\bar{\mathcal{D}}_a \chi_a - 2 \langle \chi_f H^\dagger \rangle_G|^2 \right] \\ &= \int_{\mathbb{C}} \text{Tr} \left[ |\mathcal{D}_f \chi_f|^2 + |\bar{\mathcal{D}}_a \chi_a|^2 + |\chi_a H|^2 + 4 |\langle \chi_f H^\dagger \rangle_G|^2 + c \right] , \end{aligned} \quad (3.35)$$

where we have set  $e^2 = 4$  and the cross term  $c$  has been rewritten into two total derivative terms by using the BPS equation  $\bar{\mathcal{D}}H = 0$ :  $c = \text{Tr} \left[ i\partial \left( \chi_f H^\dagger \chi_a^\dagger \right) - i\bar{\partial} \left( \chi_a H \chi_f^\dagger \right) \right]$ . Since we are interested in normalizable physical zero-modes, we require the fluctuations to vanish at spatial infinity ( $|z| \rightarrow \infty$ ). Thus the total derivative terms vanish. We now want to show  $\chi_a = 0$  and  $\chi_f = 0$  from the conditions

$$\mathcal{D}_f \chi_f = 0, \quad \bar{\mathcal{D}}_a \chi_a = 0, \quad \chi_a H = 0, \quad \langle \chi_f H^\dagger \rangle_G = 0. \quad (3.36)$$

The  $N$ -by- $N_f$  matrix field  $H$  (background field) has the maximum rank  $N$  almost everywhere except at the vortex centers. Thus we conclude  $\chi_a = 0$  from  $\chi_a H = 0$ . For  $\chi_f$  we need a more elaborated argument except for the case of  $G = U(N)$  and  $N_f = N$  [17]. Let us explain  $\chi_f = 0$  for the case of  $G' = SU(N)$  and  $N_f > N$ .<sup>9</sup> First, we solve  $\bar{\mathcal{D}}\chi_f^\dagger = 0$  by a combination of the background field  $S$  and a holomorphic matrix function  $\tilde{H}_0(z)$  by  $\chi_f^\dagger = \tilde{H}_0(z)S$ . Then consider a holomorphic  $G' = SU(N)$  invariant operator (i.e. an  $N_f$ -by- $N_f$  matrix)  $M \equiv \chi_f^\dagger H = \tilde{H}_0(z)H_0(z)$ . Note that  $\chi_f^\dagger$  can be reconstructed from  $M$ , because  $H$  has the maximal rank in the broken (Higgs) vacuum. However,  $M$  vanishes at infinity because  $\chi_f$  should be normalizable, hence we require  $\tilde{H}_0(z)H_0(z) \rightarrow 0$ . The holomorphic matrix  $\tilde{H}_0(z) = 0$  is a unique solution. Thus we conclude  $\chi_f = 0$ . For the generic simple group  $G'$ , similar arguments were done in Ref. [47].

Now we have confirmed that  $\mathbb{D}^\dagger$  has no normalizable zero-modes. So the index given in Eq. (3.29) indeed counts the correct number of the physical zero-modes.

### 3.3 Chern-Simons-Higgs theory

#### 3.3.1 The index for Chern-Simons vortices

Next we study the index of the BPS system in the non-Abelian CS theory. For simplicity, we set  $\kappa = \mu$ , hence we will be using Eqs. (2.42) and (2.46). To the best of our knowledge, the index theorem for the CS vortices has been made only in the Abelian case in Refs. [9, 11]. With the formula (3.23), we can easily calculate the index also in the non-Abelian case with a generic gauge group  $G'$ . The fluctuations of the BPS equations read to linear order

$$i\bar{\mathcal{D}}\delta H - \delta\bar{A}H = 0, \quad (3.37)$$

$$\begin{aligned} i\bar{\mathcal{D}}\delta A - i\mathcal{D}\delta\bar{A} &= \frac{16\pi^2}{\kappa^2} \left\langle \langle \delta H H^\dagger \rangle_G H H^\dagger + \langle H H^\dagger \rangle_G \delta H H^\dagger - \frac{v^2}{2} \delta H H^\dagger \right\rangle_G \\ &\quad + \frac{16\pi^2}{\kappa^2} \left\langle \langle H \delta H^\dagger \rangle_G H H^\dagger + \langle H H^\dagger \rangle_G H \delta H^\dagger - \frac{v^2}{2} H \delta H^\dagger \right\rangle_G. \end{aligned} \quad (3.38)$$

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<sup>9</sup>We thank K. Ohashi for this point.



We choose the gauge fixing condition as

$$i\bar{\mathcal{D}}\delta A + i\mathcal{D}\delta\bar{A} = -\frac{16\pi^2}{\kappa^2} \left\langle \langle \delta H H^\dagger \rangle_G H H^\dagger + \langle H H^\dagger \rangle_G \delta H H^\dagger - \frac{v^2}{2} \delta H H^\dagger \right\rangle_G \\ + \frac{16\pi^2}{\kappa^2} \left\langle \langle H \delta H^\dagger \rangle_G H H^\dagger + \langle H H^\dagger \rangle_G H \delta H^\dagger - \frac{v^2}{2} H \delta H^\dagger \right\rangle_G. \quad (3.39)$$

Combining Eqs. (3.38) and (3.39), we obtain

$$i\mathcal{D}\delta\bar{A} = -\frac{16\pi^2}{\kappa^2} \left\langle \langle \delta H H^\dagger \rangle_G H H^\dagger + \langle H H^\dagger \rangle_G \delta H H^\dagger - \frac{v^2}{2} \delta H H^\dagger \right\rangle_G. \quad (3.40)$$

Taking the transpose of the Eqs. (3.37) and (3.40), we can now rewrite the above linear equations as the Dirac-type equation (3.1) with the operator  $\mathbb{D}$  defined as

$$\mathbb{D} = \begin{pmatrix} i\bar{\mathcal{D}}_f^T & -H^T \\ \frac{16\pi^2}{\kappa^2} \left\langle \bar{H} H^T \langle \bar{H} \circ \rangle_G + \bar{H} \circ \langle \bar{H} H^T \rangle_G - \frac{v^2}{2} \bar{H} \circ \right\rangle_G & i\mathcal{D}_a^T \end{pmatrix}. \quad (3.41)$$

Although this Dirac operator takes a somewhat complicated form, it is indeed in the class of operators (3.18). Thus we can immediately apply the result of Sec. 3.1 also to this case and hence the index is given by Eq. (3.24).

Let us explain the result for each type of vacuum. First for the topological solitons the result is very sound and the index is exactly

$$\mathcal{I} = \hat{\mathcal{I}} = N_f N \nu = \frac{N_f N k}{n_0}. \quad (3.42)$$

Note that the index is always an integer and this result depends on the gauge group  $G'$  only through  $n_0$ . We will see that this index indeed counts the number of zero-modes.

In the case of the non-topological solitons in the unbroken vacuum, the magnetic flux is not quantized, so  $\hat{\mathcal{I}}$  is not an integer

$$\hat{\mathcal{I}} = \frac{N_f N (k' + \alpha)}{n_0}, \quad (3.43)$$

with  $k' \in \mathbb{Z}_{\geq 0}$  and  $\alpha \in \mathbb{R}_+$ . However, since the index  $\mathcal{I}$  counts the number of the zero-modes, it should give an integer. In order to obtain the correct result, we have to subtract contributions from the continuum due to unbounded resonances [50]. Thus the correct index is given by

$$\mathcal{I} = \frac{N_f N (k' + \hat{\alpha})}{n_0}, \quad (3.44)$$

where  $\hat{\alpha}$  is the largest integer less than  $\alpha$  ( $\alpha - \hat{\alpha} < 1$ ).  $k'$  is the number of topological solitons living “inside” the non-topological solitons and the number is limited by the amount of non-topological winding  $\alpha$ . Note that this index still includes some non-physical zero-modes which change the magnetic flux [9]. We will return this point later.

When the topological and non-topological solitons coexist, the index is mixed as

$$\mathcal{I} = \frac{N_f N k}{n_0} + \frac{N_f N (k' + \hat{\alpha})}{n_0}, \quad (3.45)$$

where the first term is related to the topological solitons while the second term is related to the non-topological solitons. Here we intend that the  $k$  vortices are topological in the (partially) broken vacuum, while the  $k'$  vortices are topological vortices living “inside” the non-topological vortices in the (partially) unbroken vacuum.

### 3.3.2 Vanishing theorem for Chern-Simons vortices

Next, we would like to make a vanishing theorem (3.30) for  $\mathbb{D}^\dagger$  in the CS case. The adjoint operator reads

$$\mathbb{D}^\dagger = \begin{pmatrix} i\mathcal{D}_f^T & \frac{8\pi^2}{\kappa^2} H^T \left( \langle \bar{H} H^T \circ \rangle_G + \circ \langle \bar{H} H^T \rangle_G - \frac{v^2}{2} \right) \\ -2 \langle \bar{H} \circ \rangle_G & i\bar{\mathcal{D}}_a^T \end{pmatrix}. \quad (3.46)$$

The question is if there exist normalizable zero-modes  $\chi$

$$\mathbb{D}^\dagger \chi = 0, \quad \chi = \begin{pmatrix} \chi_f^T \\ \chi_a^T \end{pmatrix}, \quad (3.47)$$

where  $\chi_f^T$  is an  $N_f$ -by- $N$  matrix and  $\chi_a^T$  is an  $N$ -by- $N$  matrix. Unfortunately, we cannot use the same trick as we used in Eq. (3.35) since the cross terms associated with the completion of the square do not vanish even by the use of the BPS equations. Therefore, we have to solve the problem in a more straightforward way. Let us write down the Dirac equation as follows

$$\mathcal{D}_f \chi_f = \frac{i8\pi^2}{\kappa^2} \left( \langle \chi_a H H^\dagger \rangle_G + \langle H H^\dagger \rangle_G \chi_a - \frac{v^2}{2} \chi_a \right) H, \quad (3.48)$$

$$\bar{\mathcal{D}}_a \chi_a = -i2 \langle \chi_f H^\dagger \rangle_G, \quad (3.49)$$

Taking the adjoint derivative  $\mathcal{D}_a$  of the second equation, and using the BPS equations, allows us to combine the two equations as

$$\mathcal{D}_a \bar{\mathcal{D}}_a \chi_a = \frac{16\pi^2}{\kappa^2} \left\langle \left( \langle \chi_a H H^\dagger \rangle_G + \langle H H^\dagger \rangle_G \chi_a - \frac{v^2}{2} \chi_a \right) H H^\dagger \right\rangle_G. \quad (3.50)$$

This is a non-Abelian Schrödinger equation for  $\chi_a$  with the background configuration  $A_i$  and  $H$ . This tells us that  $\mathbb{D}^\dagger$  has a non-zero kernel if and only if the Schrödinger equation has a normalizable zero-eigenstate.

Unfortunately, up to now, no proofs have been given even in the simplest Abelian case but  $\chi_a = 0$  has been expected in general not to have normalizable zero-modes [9, 11]. For the non-Abelian cases, we have nothing to add beyond the Abelian case. We would in general expect that  $\chi_a = 0$  and  $\chi_f = 0$  holds and that the index (3.24) counts the complex dimension of the kernel of  $\mathbb{D}$  and hence the number of bosonic zero-modes of the BPS configuration. A partial evidence for us to expect this is that we will explicitly realize the same number of the physical zero-modes for the topological solitons as is given by the index theorem result (3.42).

## 4 The moduli matrix method

In Sec. 3, we have demonstrated the existence of a certain number of zero-modes for the BPS solutions by using a formal mathematical computation of the index of a Dirac operator describing the BPS configurations under consideration. Here in this section, on the contrary, we will realize the zero-modes in a more explicit manner. The so-called moduli matrix method [28, 31, 30, 20] is suitable for that purpose which was first developed to describe the topological solitons in the Higgs phase of supersymmetric YM theories without CS terms.

Let us apply the moduli matrix for the vortex system [30, 20, 51] in order to solve the first of the BPS equations (2.16) and (2.42) (which is the same in both the YM and CS theories):

$$H = S^{-1}H_0(z) , \quad \bar{A} = -iS^{-1}\bar{\partial}S , \quad S \in G^{\mathbb{C}} , \quad (4.1)$$

where  $H_0(z)$  is an  $N$ -by- $N_f$  complex matrix which is holomorphic in  $z$ .  $H_0(z)$  is called the moduli matrix [28, 31, 30, 20] and the decomposition of the gauge field is called the Karabali-Nair form [52].  $S$  takes a value in the complexified gauge group  $G^{\mathbb{C}}$ . For instance for  $G = U(N)$  the complexified gauge symmetry is  $GL(N, \mathbb{C})$ . Note that the one-form  $\bar{A}$  is not pure gauge. The gauge symmetry and the flavor symmetry act on the new matrices as follows

$$(S^{-1}, H_0) \rightarrow (U_c S^{-1}, H_0 U_f) . \quad (4.2)$$

Furthermore, there is an equivalence relation, which is denoted  $V$ -equivalence and it acts as

$$(S, H_0) \sim V(z)(S, H_0) , \quad V(z) \in G^{\mathbb{C}} , \quad (4.3)$$

with all the elements in  $V(z)$  being any holomorphic function with respect to  $z$ . In order not to change the winding number (energy) of the solution, we should impose that the determinant of  $V(z)$  is non-zero.

That is all for the first equation. The moduli matrix  $H_0(z)$  has all the information about the topological solitons in the YM theories [31, 30]. It is conjectured in Ref. [26] that the same holds for the topological solitons in the (non-Abelian) CS theories.

Since the classification of the moduli space for the non-Abelian vortices has been carried out in the case of YM theories, we will not repeat it here. In the case of CS theories, the results of Ref. [26] claim that the moduli spaces of the YM vortices apply also to the Higgs phase of the CS theories. However, in the case of CS theory – as already mentioned in Sec. 2.2 – there exist not only the Higgs phase but also the CS phase (unbroken phase) and the partially broken phases. In these cases, we cannot use the arguments of Ref. [26] and furthermore, as we will see later, the moduli matrix is not sufficient to describe all of the moduli parameters possessed by the vortices.

After introducing the  $S$  field in Eq. (4.1), it is natural to introduce a local gauge invariant field [31, 30, 20]

$$\Omega = SS^\dagger, \quad \Omega = \omega \hat{\Omega}. \quad (4.4)$$

Note that the complex matrix field  $S$  can be decomposed as  $S = s\hat{S}$  with the Abelian part  $s \in U(1)^\mathbb{C} \sim \mathbb{C}^\star$  and the non-Abelian part  $\hat{S} \in G^\mathbb{C}$ . In the right equation above, we have defined  $\omega = |s|^2$  and  $\hat{\Omega} = \hat{S}\hat{S}^\dagger$ . We will see that this  $\Omega$  will play a central role in the following sections. Note that  $\Omega$  transforms as  $\Omega \rightarrow V\Omega V^\dagger$  with respect to the  $V$ -transformation. Let us also introduce a “current” defined as [52]

$$\mathcal{J} \equiv \partial\Omega\Omega^{-1}. \quad (4.5)$$

This  $\mathcal{J}$  transforms as a holomorphic connection under the  $V$ -transformation

$$\mathcal{J} \rightarrow V\mathcal{J}V^{-1} + \partial VV^{-1}. \quad (4.6)$$

In terms of  $\mathcal{J}$ , the magnetic field can be rewritten as

$$F_{12} = -2S^{-1}\bar{\partial}JS, \quad (4.7)$$

(with  $F_{12}^0 = -2\sqrt{2N}\bar{\partial}\partial\log\omega$  and  $\hat{F}_{12} = -2\hat{S}^{-1}\bar{\partial}(\partial\hat{\Omega}\hat{\Omega}^{-1})\hat{S}$ ). The magnetic flux is expressed as

$$\Phi = -\int_{\mathbb{C}} \text{Tr}[F_{12}] = 2\text{Tr} \int_{\mathbb{C}} \bar{\partial}\mathcal{J} = 2N \int_{\mathbb{C}} \bar{\partial}\partial\log\omega, \quad (4.8)$$

while the  $U(1)$  winding number is given by

$$\nu = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\partial\log\omega. \quad (4.9)$$

Let us define a covariant derivative with the connection  $i\mathcal{J}$ . For example, for an adjoint field  $\phi$ , transforming as  $\phi \rightarrow V\phi V^{-1}$ , then we have

$$\mathcal{D}_v\phi = \partial\phi - [\mathcal{J}, \phi] . \quad (4.10)$$

$\mathcal{D}_v\phi$  transforms homogeneously under the  $V$ -transformation.

## 4.1 Yang-Mills-Higgs theory

For the readers who are not familiar with the moduli matrix, let us review several examples (the details of this topic can be found in Ref. [20]). As the first example, let us review the Abelian vortex. The moduli matrix for  $k$  vortices reads

$$H_0(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{k-1} z^{k-1} + z^k = \prod_{i=1}^k (z - z_i) , \quad (4.11)$$

where we have fixed the coefficient of  $z^k$  to unity by using the  $V$ -equivalence relation. The set of complex parameters  $\{a_0, a_1, \dots, a_{k-1}\}$  are the free parameters (the position moduli) of the BPS solutions. Hence, the moduli space of the  $k$  Abelian vortices is

$$\mathcal{M}_{U(1)}^k = \mathbb{C}^k . \quad (4.12)$$

The complex dimension is  $k$ , which is consistent with the index theorem result (3.24).

The second example we will review here is the  $k = 1$  vortex in the  $G' = SU(N)$  model with  $N_f = N$  flavors. After fixing the  $V$ -equivalence, one can obtain the following matrix

$$H_0(z) = \begin{pmatrix} 1 & & b_1 \\ & \ddots & \vdots \\ & & 1 & b_{N-1} \\ & & & z - z_1 \end{pmatrix} . \quad (4.13)$$

The set of complex parameters  $\{b_1, \dots, b_{N-1}\}$  corresponds to the inhomogeneous coordinate of the orientational zero-modes  $\mathbb{C}P^{N-1}$  while  $z_0$  is the position (translational) zero-mode. Hence, we obtain the corresponding moduli space

$$\mathcal{M}_{U(N)}^{k=1} = \mathbb{C} \times \mathbb{C}P^{N-1} . \quad (4.14)$$

The complex dimension is  $N$  which is again consistent with the index theorem (3.24). In this way, the moduli matrix provides us a realization of the physical zero-modes.

We would like to emphasize that the use of the moduli matrix is quite easy. One just needs to choose a holomorphic matrix which satisfies the conditions posed on the invariants with respect to  $G'$  and it should be consistent with the chosen boundary conditions. For instance, any matrix  $H_0(z)$  whose determinant is a polynomial in  $z$  of degree  $k$ , generates  $k$  non-Abelian vortices in the case of  $G' = SU(N)$ . When we choose the form of the moduli matrix  $H_0(z)$  we should fix the  $V$ -equivalence in order to eliminate unphysical complex parameters. Once we have done this, one can easily read off the physical moduli parameters for the solutions. For a generic gauge group  $G'$ , we have to take further conditions into account. We will not discuss those details here as they can be found in Refs. [46, 47].

In order to solve completely the BPS equations, we now have to solve the second equation of the BPS equations (2.16). Unfortunately, this is not an easy task since no analytic solutions have been obtained even in the case of the simplest Abelian models. Solving the second equation is very important in the following two senses:

- 1) We have to confirm the existence of the BPS solution for each given moduli matrix. If the solution does not exist, it means that the complex parameters in the moduli matrix cannot be moduli parameters.
- 2) We have to confirm uniqueness of the solutions of the second equation. If we are not able to show the uniqueness, it means that there might exist additional zero-modes which are not included in the moduli matrix description.

These problems have been resolved only in the case of the  $k$  vortices in the Abelian-Higgs model by Taubes [53]. But there is no available proof for the non-Abelian cases<sup>10</sup>.

The strongest statement in favor of our belief of existence of the solutions in the non-Abelian models is that several numerical solutions have been found explicitly in the literature. Hence, we are quite confident that the existence holds in all our models, although we currently do not have a rigorous mathematical proof.

On the uniqueness problem, we have observed that the number of the physical complex parameters in the moduli matrix coincides with that of the index theorem (3.24) in all known models [17, 47]. Although this coincidence is in favor of our intuition, we still miss a direct relation between the uniqueness problem and the index theorem. We will partially solve the uniqueness problem of the non-Abelian vortices in YM and CS theories in Sec. 5.

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<sup>10</sup>This is true on the infinite  $\mathbb{C}$ -plane. See however Ref. [54] for non-Abelian  $U(N)$  vortices on compact Riemann surfaces.

Let us rewrite the second equation of (2.16) in terms of  $\Omega$  and the moduli matrix  $H_0(z)$

$$\bar{\partial}\partial \log \omega = \frac{e^2}{4N} \text{Tr} [v^2 \mathbf{1}_N - \Omega_0 \Omega^{-1}] , \quad (4.15)$$

$$\bar{\partial}(\partial \hat{\Omega} \hat{\Omega}^{-1}) = -\frac{g^2}{2} \langle \Omega_0 \Omega^{-1} \rangle_{G'} , \quad (4.16)$$

with  $\Omega_0 \equiv H_0 H_0^\dagger$  and  $\Omega = \omega \hat{\Omega}$ . We have used the following relation

$$\langle X \rangle_{G'} = \text{Tr} [X t^a] t^a = \text{Tr} [X \tilde{t}^a] \tilde{t}^a , \quad \tilde{t}^a \equiv \hat{S} t^a \hat{S}^{-1} = S t^a S^{-1} , \quad (4.17)$$

where  $\tilde{t}^a$  is an automorphism of  $t^a$ . These equations are called the master equations for the YM vortices. The equal coupling case,  $g = e$  can for a generic group neatly be written as

$$\bar{\partial}(\partial \Omega \Omega^{-1}) = \frac{e^2}{2} \langle v^2 \mathbf{1} - \Omega_0 \Omega^{-1} \rangle_G . \quad (4.18)$$

Before closing this section, let us see the asymptotic behavior of  $\Omega$ . It is again sufficient to consider the minimal winding solution in the Abelian model. Hence we take  $H_0(z) = z$  and then  $\omega$  approaches  $\Omega_0/v^2 = |z|^2/v^2$ . Plugging this into the master equation, one easily obtains the asymptotic solution

$$\omega = \frac{|z|^2}{v^2} [1 + q K_0(m_e |z|)] , \quad (4.19)$$

where  $K_0$  is the modified Bessel function of the second kind and  $q$  is an unknown constant parameter which can be determined numerically.

## 4.2 Chern-Simons-Higgs theory

As mentioned above, the first BPS equation is common for the YM models and the CS models. Hence, the moduli matrix method which solves the first BPS equation explained in Sec. 4.1 can be applied to the CS models without any modifications [26]. This is one of the significant features of the moduli matrix method, namely its application range is indeed quite large.

Since we have already solved the first equation, the remaining task is to solve the second equation of (2.42). In Ref. [26], the BPS equations were given for a generic gauge group  $G$ , but the master equations were provided only for  $G' = SU(N), SO(N), USp(2M)$ . Here we will provide the master equations completely independent of the choice of  $G'$

$$\bar{\partial}\partial \log \omega = \frac{4\pi^2}{N^2 \kappa^2} \text{Tr} [v^2 \mathbf{1}_N - \Omega_0 \Omega^{-1}] \text{Tr} [\Omega_0 \Omega^{-1}] - \frac{8\pi^2}{N \kappa \mu} \text{Tr} [\langle \Omega_0 \Omega^{-1} \rangle_{G'} \Omega_0 \Omega^{-1}] , \quad (4.20)$$

$$\bar{\partial}(\partial \hat{\Omega} \hat{\Omega}^{-1}) = \frac{8\pi^2}{N \kappa \mu} \text{Tr} [v^2 \mathbf{1}_N - \Omega_0 \Omega^{-1}] \langle \Omega_0 \Omega^{-1} \rangle_{G'} - \frac{16\pi^2}{\mu^2} \langle \langle \Omega_0 \Omega^{-1} \rangle_{G'} \Omega_0 \Omega^{-1} \rangle_{G'} . \quad (4.21)$$

The equal coupling case,  $\mu = \kappa$  can neatly be written for a generic group as

$$\bar{\partial}(\partial\Omega\Omega^{-1}) = \frac{16\pi^2}{\kappa^2} \langle \langle v^2 \mathbf{1}_N - \Omega_0 \Omega^{-1} \rangle_G \Omega_0 \Omega^{-1} \rangle_G . \quad (4.22)$$

These are called the master equations for the CS theory.

Concerning the master equation, we have the same problems as in the YM theories. Namely, the existence and uniqueness of the solutions. To the former problem, we are in the same situation as in the case of YM theories, the best argument in favor of the existence we have currently, is the various numerical solutions to the non-Abelian CS BPS equations [26]. The existence of the solutions however is an important future problem but it is beyond the scope of this paper. In the Abelian case, the existence however has been proved in the topological case [55] and in the non-topological case (for radially symmetric solutions) [56].

As for the uniqueness problem, we will partially solve it in the Sec. 5. But there is a big difference between the YM theories and the CS theories. As we will see in Sec. 5, the solutions to the master equation have their own moduli parameters, when we choose a vacuum different from the Higgs vacuum at the boundary.

#### 4.2.1 Abelian Chern-Simons solitons

We will now briefly describe the Abelian solutions [9] in the moduli matrix formalism with  $N_f = 1$  flavor. Starting with the vacuum configurations. There are two vacua: i) the broken vacuum and ii) the unbroken vacuum. The Higgs field is  $H = v$  (up to  $U(1)$  gauge symmetry) in the former case, so it is described by

$$\text{broken vacuum : } H_0 = 1 , \quad s^{-1} = v . \quad (4.23)$$

Note that the  $V$ -equivalence has been fixed by the choice  $H_0 = 1$ . In the unbroken vacuum,  $H = 0$ , we choose the moduli matrix as follows

$$\text{unbroken vacuum : } H_0 = 0 , \quad s^{-1} = 1 , \quad (4.24)$$

Here we have set  $s = 1$  by using the  $V$ -equivalence relation. At first glance, the  $U(1)$  gauge symmetry seems to be broken since  $s^{-1}$  transforms as  $s^{-1} \rightarrow g s^{-1}$  with  $g \in U(1)$ . But this transformation can be absorbed by an according  $V$ -transformation, such that no symmetries are broken.

Let us next consider  $k$  topological vortices. It is generated by the moduli matrix  $H_0 = \prod_{i=1}^k (z - z_i)$ . The number of moduli parameters is  $k$  and is in accord with the index (3.24). The



master equation determines  $\omega$  as

$$\bar{\partial}\partial \log \omega = \frac{4\pi^2}{\kappa^2} \prod_{i=1}^k |z - z_i|^2 \omega^{-1} \left( v^2 - \prod_{i=1}^k |z - z_i|^2 \omega^{-1} \right). \quad (4.25)$$

We impose the boundary condition for  $\omega$  in such a way that the Higgs field approaches to the Higgs phase  $|H| \rightarrow v$ . With respect to  $\omega$ , this is equal to imposing the boundary condition

$$\omega \rightarrow v^{-2} \prod_{i=1}^k |z - z_i|^2, \quad \text{as } |z| \rightarrow \infty. \quad (4.26)$$

Note that this boundary condition is unique for obtaining a regular solution. We would like to stress that all the moduli parameters of the topological vortices are included in the moduli matrix. Unfortunately, no analytic solutions to this equation even for  $k = 1$  are known. The asymptotic solution of  $k = 1$  however is

$$\omega_{k=1} = v^{-2} |z|^2 [1 + \tilde{q} K_0(m_{\kappa\infty}|z|)] , \quad (4.27)$$

where  $\tilde{q}$  is an unknown parameter which can be determined numerically. There is no difference in the asymptotic form of the topological solitons of the Abelian-Higgs model and the CS model, see Eqs. (4.19) and (4.27). This is because they are topological solitons in the broken vacuum. A tiny difference lies in the masses of the Higgs fields  $m_e$  versus  $m_{\kappa\infty}$  and in the numerical coefficients  $q$  versus  $\tilde{q}$ .

Let us next consider the non-topological vortex with  $k$  Higgs zeros and magnetic flux  $\Phi = -\int_{\mathbb{C}} F_{12} = 2\pi(k + \alpha)$ . The total magnetic flux is not necessarily an integer. In fact,  $\alpha$  can be an arbitrary real number. We again take the moduli matrix  $H_0 = \prod_{i=1}^k (z - z_i)$ . Then the master equation is also the same as Eq. (4.25). We solve this with a different boundary condition for  $\omega$  consistent with  $H \rightarrow 0$ . It is determined by the total amount of magnetic flux

$$F_{12} = -2\bar{\partial}\partial \log \omega \rightarrow -2\bar{\partial}\partial \log |z|^{2(k+\alpha)}, \quad (4.28)$$

for  $|z| \rightarrow \infty$ . Thus the desired boundary condition reads

$$\omega \rightarrow C^{-1} |z|^{2\alpha} \prod_{i=1}^k |z - z_i|^2, \quad (4.29)$$

where  $C$  is a unique numerical constant.  $H$  asymptotically approaches the unbroken vacuum as  $H \rightarrow C/|z|^\alpha$ . In this way, we have found  $k$  zero-modes  $\{z_i\}$  in the moduli matrix as in the case of topological vortices. However, in Sec. 5.2, we will see that Eq. (4.29) is not the most generic boundary condition on  $\omega$  for the soliton solutions with a fixed magnetic flux  $\Phi = 2\pi(k + \alpha)$ . We will also see that there exist additional zero-modes which are not accounted for in the moduli matrix but on the other hand reside in  $\omega$ .

#### 4.2.2 $U(2)$ Chern-Simons solitons

Let us next explain the solitons in the  $U(2)$  CS theory ( $\kappa = \mu$ ) with  $N_f = 2$  in terms of the moduli matrix.

We start by describing the vacua as we did in the Abelian case above. As we have explained in Sec. 2.2.2, there are three vacua in the case at hand: (2) the fully broken vacuum, (1) the partially broken vacuum and (0) the unbroken vacuum. The Higgs fields corresponding to these vacua are given in Eq. (2.53), while the corresponding moduli matrices are given by

$$H_0^{(2)} = v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_0^{(1)} = v \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \quad H_0^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.30)$$

Here  $\alpha, \beta \in \mathbb{C}$  parametrize the vacuum moduli space i.e.  $\mathbb{CP}^1$ . We fix these parameters as  $(\alpha, \beta) = (1, 0)$  by using  $V$ -equivalence and flavor symmetry. For all the vacuum states,  $S$  is chosen to be the unit matrix.

Let us next consider the solitons. Since we are working with  $\kappa = \mu$ , most solutions in the  $U(2)$  case can be obtained by the embedding of some Abelian solutions. For instance, the  $k = 1$  topological vortex can be obtained by

$$H_0 = \begin{pmatrix} z - z_0 & 0 \\ b & 1 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \frac{\omega_s + |b|^2 |z - z_0|^2}{1 + |b|^2} & b^*(z - z_0) \\ b(z - z_0)^* & 1 + |b|^2 \end{pmatrix}, \quad (4.31)$$

where  $\omega_s$  is the  $k = 1$  solution of the Abelian master equation (4.25) with the boundary condition  $\omega_s \rightarrow v^{-2}|z|^2$ . The  $U(1)$  winding number is not integer but half-integer  $\nu = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \partial \log \omega_s^{\frac{1}{2}} = 1/2$ . We have realized two complex moduli parameters  $z_0$  and  $b$  in the moduli matrix. The former is the position modulus and the latter is the orientational modulus parameter of  $\mathbb{CP}^1$ , see Ref. [32] for details. Hence, we have two moduli ( $N_f N \nu = 2$ ) which is in accord with the index result (3.24).

Next we study solitons in the partially broken vacuum. We are interested in the semi-local vortex in this vacuum. The moduli matrix and  $\Omega$  are respectively given by

$$H_0 = \begin{pmatrix} z - z_0 & a \\ 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega_s & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.32)$$

where  $z_0$  again denotes the position. The complex parameter  $a$  is a combination of the transverse size  $|a|$  and a relative phase  $\text{Arg } a$ , while  $\omega_s$  is the solution to the master equation

$$\bar{\partial} \partial \log \omega_s = \frac{4\pi^2}{\kappa^2} \left[ v^2 - \frac{|z - z_0|^2 + |a|^2}{\omega_s} \right] \frac{|z - z_0|^2 + |a|^2}{\omega_s}. \quad (4.33)$$

This equation should be solved with the boundary condition

$$\omega_s \rightarrow \frac{|z - z_0|^2 + |a|^2}{v^2} . \quad (4.34)$$

Thus, the  $U(1)$  winding number is again a half integer  $\nu = \frac{1}{\pi} \int_{\mathbb{C}} \partial \bar{\partial} \log \omega_s^{\frac{1}{2}} = 1/2$ . Hence, we find again two moduli parameters in accord with the index result (3.24). In the Abelian-Higgs model, the zero-mode  $a$  of the semi-local vortex is a non-normalizable zero-mode. We suspect that  $a$  in the semi-local CS soliton is also non-normalizable.

One can also consider the semi-local non-topological solitons in the partially unbroken vacuum. We do not work it out in this paper, although it might be interesting. The reader who is interested in this topic can easily extend our moduli matrix method to this case by comparing our method and the results in Ref. [13].

Finally, we study the non-topological solitons in the fully unbroken vacuum. To this end, we choose the following moduli matrix and  $\Omega$

$$H_0 = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} , \quad \Omega = \begin{pmatrix} (1 + |b|^2)\omega_s & 0 \\ 0 & 1 \end{pmatrix} , \quad (4.35)$$

where  $\omega_s$  satisfies the master equation (4.25) with  $k = 0$ . Hence  $\omega_s$  behaves asymptotically like  $\sim |z|^{2\alpha}$ . The  $U(1)$  winding number is  $\nu = \frac{1}{\pi} \int_{\mathbb{C}} \partial \bar{\partial} \log \omega_s^{\frac{1}{2}} = \alpha/2$ , which is a half of that of the Abelian case. The complex parameter  $b$  is a realization of the internal orientational moduli  $\mathbb{CP}^1 \simeq SU(2)/U(1)$  which we have explained in Sec. 2.2.2. Suppose that the Abelian solution has  $\hat{\alpha} = 2$  (minimal choice) and hence provides one modulus parameter, then we have found two moduli parameters in accord with the index result (3.24), see however Sec. 5 for more comments on this.

#### 4.2.3 Higher rank gauge group: $U(4)$

Let us next highlight the vortices in the higher rank gauge group. For concreteness, we take  $G = U(4)$  in this section with  $N_f = 4$  flavors. We will explain the vortices not in the full Higgs phase but in the intermediate and unbroken vacua since the latter ones have not been studied in literature.

Let us first consider the following intermediate vacuum

$$H = v \text{ diag}(1, 1, 0, 0) , \quad (4.36)$$

where  $U(4)_c \times SU(4)_f$  is broken to  $U(2)_c \times S[U(2)_f \times U(2)_{c+f}]$ . We can put both the topological and non-topological solitons in this vacuum. For instance, the minimal winding ( $k = 1$ ) topological

vortex can be generated by the moduli matrix

$$\Omega = v^{-2} \text{diag}(\omega_1, \omega_2, \omega_3, \omega_4) , \quad H_0 = \text{diag}(z, 1, 0, 0) . \quad (4.37)$$

The corresponding master equations are of the form

$$\partial \bar{\partial} \log \omega_1 = \frac{4\pi^2 v^4}{\kappa^2} (1 - |z|^2 \omega_1^{-1}) |z|^2 \omega_1^{-1} , \quad (4.38)$$

$$\partial \bar{\partial} \log \omega_2 = \frac{4\pi^2 v^4}{\kappa^2} (1 - \omega_2^{-1}) \omega_2^{-1} , \quad (4.39)$$

$$\partial \bar{\partial} \log \omega_3 = 0 , \quad (4.40)$$

$$\partial \bar{\partial} \log \omega_4 = 0 . \quad (4.41)$$

The first equation is exactly the same as the master equation in the Abelian case. We solve it with the boundary condition

$$\omega_1 \rightarrow |z|^2 , \quad |z| \rightarrow \infty . \quad (4.42)$$

The second one is solved by  $\omega_2 = 1$  and the last two can be solved as  $\omega_{3,4} = \text{const.}$  We can set the constants to one using a  $V$ -transformation. Thus we find

$$\Omega = v^{-2} \text{diag}(\omega_1, 1, 1, 1) \rightarrow v^{-2} \text{diag}(|z|^2, 1, 1, 1) . \quad (4.43)$$

Let us decompose  $\Omega$  into the Abelian and non-Abelian parts

$$\Omega = v^{-2} \omega_1^{\frac{1}{4}} \text{diag} \left( \omega_1^{\frac{3}{4}}, \omega_1^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}} \right) . \quad (4.44)$$

From this, we can read off the  $U(1)$  winding number and the magnetic flux

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \omega_1^{\frac{1}{4}} = \frac{1}{4} , \quad (4.45)$$

$$\Phi = 2\pi \nu \times 4 = 2\pi . \quad (4.46)$$

The moduli matrix in Eq. (4.37) and the corresponding master equations (4.38)–(4.41) can generate other vortices in the other vacua. If we solve Eqs. (4.38)–(4.41) with the boundary condition for  $\omega_1$

$$\omega_1 \rightarrow \frac{|z|^{2+2\alpha}}{C} , \quad \text{as } |z| \rightarrow \infty , \quad (4.47)$$

with  $\omega_{2,3,4} = 1$ , corresponding still to  $k = 1$ , it however leads to the non-topological soliton in the intermediate vacuum

$$H = v \text{diag}(0, 1, 0, 0) . \quad (4.48)$$

Its  $U(1)$  winding number and magnetic flux are

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \omega_1^{\frac{1}{4}} = \frac{1+\alpha}{4} , \quad (4.49)$$

$$\Phi = 2\pi\nu \times 4 = 2\pi(1+\alpha) . \quad (4.50)$$

Furthermore, if we solve Eqs. (4.38)–(4.41) with the boundary condition for  $\omega_{1,2}$

$$\omega_1 \rightarrow |z|^2 , \quad \omega_2 \rightarrow \frac{|z|^{2\beta}}{C'} , \quad |z| \rightarrow \infty , \quad (4.51)$$

we have the topological vortex with the minimal winding number ( $k=1$ ) and the non-topological soliton in the different intermediate vacuum

$$H = v \operatorname{diag}(1, 0, 0, 0) . \quad (4.52)$$

In this case,  $\Omega$  is decomposed as

$$\Omega = v^{-2} \omega_1^{\frac{1}{4}} \omega_2^{\frac{1}{4}} \operatorname{diag} \left( \omega_1^{\frac{3}{4}} \omega_2^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}} \omega_2^{\frac{3}{4}}, \omega_1^{-\frac{1}{4}} \omega_2^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}} \omega_2^{-\frac{1}{4}} \right) . \quad (4.53)$$

From this, we can read off the  $U(1)$  winding number and the magnetic flux

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \left( \omega_1^{\frac{1}{4}} \omega_2^{\frac{1}{4}} \right) = \frac{1}{4} + \frac{\beta}{4} , \quad (4.54)$$

$$\Phi = 2\pi\nu \times 4 = 2\pi + 2\pi\beta . \quad (4.55)$$

Note that, although this form is very similar to the previous example, the two configurations are completely different. In the previous example there is a non-topological soliton with a topological soliton “inside” and hence flux  $2\pi(1+\alpha)$  whereas in the case here, there are both the topological vortex with the quantized magnetic flux  $2\pi$  and the non-topological vortex with the flux  $2\pi\beta$ .

The last possibility of boundary conditions for Eqs. (4.38)–(4.41) is

$$\omega_1 \rightarrow \frac{|z|^{2+2\alpha}}{C} , \quad \omega_2 \rightarrow \frac{|z|^{2\beta}}{C'} , \quad |z| \rightarrow \infty . \quad (4.56)$$

This leads to the two non-topological solitons in the unbroken vacuum

$$H = \operatorname{diag}(0, 0, 0, 0) . \quad (4.57)$$

The solution has  $U(1)$  winding number and magnetic flux as follows

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \left( \omega_1^{\frac{1}{4}} \omega_2^{\frac{1}{4}} \right) = \frac{1+\alpha}{4} + \frac{\beta}{4} , \quad (4.58)$$

$$\Phi = 2\pi\nu \times 4 = 2\pi(1+\alpha) + 2\pi\beta . \quad (4.59)$$

#### 4.2.4 Higher rank gauge groups: $U(1) \times SO(4)$ and $U(1) \times USp(4)$

The solitons in the models with  $G' = SO(N)$  and  $G' = USp(N)$  can also easily be worked out using the moduli matrix. To be concrete, let us consider the examples  $SO(4)$  and  $USp(4)$  in this section.  $G' = SO(4), USp(4)$  can be dealt with on the same footing in the moduli matrix formalism. A tiny difference is the sign  $\epsilon = \pm 1$  of the invariant tensor  $J$  given in Eq. (2.22). In what follows, we do not need to distinguish between  $G' = SO(4)$  and  $G' = USp(4)$  since all the equations turn out to be the same in these two cases.

Let us consider the moduli matrix

$$\Omega = v^{-2} \text{diag}(\omega_1, \omega_1, \omega_2, \omega_2), \quad H_0 = \text{diag}(z^k, z^k, 1, 1). \quad (4.60)$$

With this at hand, the master equation (4.22) is simplified as

$$\begin{pmatrix} \partial \bar{\partial} \log \omega_1 \mathbf{1}_2 & \\ & \partial \bar{\partial} \log \omega_2 \mathbf{1}_2 \end{pmatrix} = \frac{4\pi^2 v^4}{\kappa^2} \begin{pmatrix} (1 - |z|^{2k} \omega_1^{-1}) |z|^{2k} \omega_1^{-1} \mathbf{1}_2 & \\ & (1 - \omega_2^{-1}) \omega_2^{-1} \mathbf{1}_2 \end{pmatrix}. \quad (4.61)$$

Note that  $\omega_1$  and  $\omega_2$  are decoupled since we imposed the special coupling relation  $\kappa = \mu$ .

The boundary conditions of  $\omega_{1,2}$  determine the type of solitons and the vacua. If we take the boundary condition

$$\omega_1 \rightarrow |z|^{2k}, \quad \text{as } |z| \rightarrow \infty, \quad (4.62)$$

with  $\omega_2 = 1$ , we get  $k$  topological (coaxial) vortices in the full Higgs phase

$$H = v \text{diag}(1, 1, 1, 1). \quad (4.63)$$

The solution is decomposed as

$$\Omega = v^{-2} \omega_1^{1/2} \text{diag}(\omega_1^{1/2}, \omega_1^{1/2}, \omega_1^{-1/2}, \omega_1^{-1/2}). \quad (4.64)$$

Thus we can read off the  $U(1)$  winding number and magnetic flux

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \omega_1^{1/2} = \frac{k}{2}, \quad (4.65)$$

$$\Phi = 2\pi\nu \times 4 = 4\pi k. \quad (4.66)$$

If we choose a different boundary condition

$$\omega_1 \rightarrow \frac{|z|^{2(k+\alpha)}}{C}, \quad \text{as } |z| \rightarrow \infty, \quad (4.67)$$

with  $\omega_2 = 1$ , it leads to the non-topological soliton in the intermediate vacuum

$$H = v \operatorname{diag}(0, 0, 1, 1) . \quad (4.68)$$

The  $U(1)$  winding number and the magnetic flux can be read off as above

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \omega_1^{1/2} = \frac{k + \alpha}{2} , \quad (4.69)$$

$$\Phi = 2\pi\nu \times 4 = 4\pi(k + \alpha) . \quad (4.70)$$

If we take a different boundary condition

$$\omega_1 \rightarrow |z|^{2k} , \quad \omega_2 \rightarrow \frac{|z|^{2\beta}}{C'} , \quad \text{as } |z| \rightarrow \infty , \quad (4.71)$$

we get  $k$  topological (coaxial) vortices as well as the non-topological vortices in the intermediate vacuum

$$H = v \operatorname{diag}(1, 1, 0, 0) . \quad (4.72)$$

The solution is decomposed as

$$\Omega = v^{-2} \omega_1^{1/2} \omega_2^{1/2} \operatorname{diag} \left( \omega_1^{1/2} \omega_2^{-1/2}, \omega_1^{1/2} \omega_2^{-1/2}, \omega_1^{-1/2} \omega_2^{1/2}, \omega_1^{-1/2} \omega_2^{1/2} \right) . \quad (4.73)$$

Thus we can read off the  $U(1)$  winding number and magnetic flux

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \left( \omega_1^{1/2} \omega_2^{1/2} \right) = \frac{k}{2} + \frac{\beta}{2} , \quad (4.74)$$

$$\Phi = 2\pi\nu \times 4 = 4\pi k + 4\pi\beta . \quad (4.75)$$

The last possible choice of boundary condition is

$$\omega_1 \rightarrow \frac{|z|^{2(k+\alpha)}}{C} , \quad \omega_2 \rightarrow \frac{|z|^{2\beta}}{C'} , \quad \text{as } |z| \rightarrow \infty . \quad (4.76)$$

This generates non-topological solitons in the unbroken vacuum

$$H = \operatorname{diag}(0, 0, 0, 0) . \quad (4.77)$$

The solution has  $U(1)$  winding number and magnetic flux as follows

$$\nu = \frac{1}{\pi} \int \partial \bar{\partial} \log \left( \omega_1^{1/2} \omega_2^{1/2} \right) = \frac{k + \alpha}{2} + \frac{\beta}{2} , \quad (4.78)$$

$$\Phi = 2\pi\nu \times 4 = 4\pi(k + \alpha) + 4\pi\beta . \quad (4.79)$$

As in the case of  $U(4)$  CS theory, the non-topological solitons in  $U(1) \times SO(4)$  and  $U(1) \times USp(4)$  carry non-Abelian zero-modes associated with the spontaneous symmetry breaking. To see this, let us consider a minimal example with the embedding solution

$$H = \text{diag}(H^{\text{NTP}}, H^{\text{NTP}}, 0, 0) \rightarrow \begin{cases} \text{diag}(v', v', 0, 0) & \text{at the origin} \\ \text{diag}(0, 0, 0, 0) & \text{at the infinity} \end{cases} . \quad (4.80)$$

The global symmetry respected at spatial infinity is  $SU(4)_f$  which however is spontaneously broken down to  $U(2)_{c+f} \times SU(2)_f$  by the soliton (the same is true for  $USp(4)$ ). Thus the non-Abelian moduli space is given by the Grassmannian

$$\mathcal{M}_{\text{orientation}} = \frac{SU(4)}{U(2) \times SU(2)} \simeq Gr_{4,2} . \quad (4.81)$$

Note that this is different from the orientational moduli  $SO(4)/U(2)$  ( $USp(4)/U(2)$ ) of the minimal topological soliton in the  $SO(4)$  ( $USp(4)$ ) model. We have observed that the topological and non-topological solitons accidentally have the same orientational moduli  $SU(N)/U(N-1) \simeq \mathbb{CP}^{N-1}$  in the  $U(N)$  CS model. This is however in general not the case.

## 5 On the uniqueness of the master equations

In this section, we will try to solve the uniqueness problem of the master equations (4.15)–(4.16) or (4.20)–(4.21), which we pointed out in Sec. 4. For that purpose, we will consider small fluctuations around the gauge invariant field  $\Omega$

$$\Omega \rightarrow \Omega_s + \delta\Omega, \quad (5.1)$$

with  $\Omega_s = S_s S_s^\dagger$  being a true solution corresponding to a given configuration  $\Omega_0$ . If the master equation has the uniqueness property<sup>11</sup>,  $\delta\Omega$  must be zero in the whole  $\mathbb{C}$ -plane. So our goal is to confirm that  $\delta\Omega = 0$ .

### 5.1 Yang-Mills-Higgs theory

In order not to introduce unessential complication to the following argument, let us consider the equal gauge coupling case  $e = g$ . The master equation in focus is Eq. (4.18). The fluctuations in

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<sup>11</sup> Since, at the boundary  $|z| \rightarrow \infty$ , the solution  $\Omega_s$  is known in terms of the lump solution for the semi-local topological solitons (which is the statement that  $\Omega_s$  is in the vacuum manifold at  $|z| \rightarrow \infty$ ) and furthermore that this solution is an algebraic solution and unique, we know that even if there exist another solution  $\Omega$  it has to obey the *same* boundary condition at  $|z| \rightarrow \infty$  and hence it follows that  $\delta\Omega \rightarrow 0$  for  $|z| \rightarrow \infty$ .



Eq. (5.1) obey to the linear order

$$\bar{\partial}\partial\phi + [\Omega_s^{-1}\bar{\partial}\Omega_s, \partial\phi] = \frac{e^2}{2}\Omega_s^{-1}\langle\Omega_0\phi\Omega_s^{-1}\rangle_G\Omega_s, \quad (5.2)$$

where we have defined the  $N$ -by- $N$  gauge invariant matrix field

$$\phi \equiv \Omega_s^{-1}\delta\Omega. \quad (5.3)$$

Note that since  $\phi^\dagger$  and  $\bar{\partial}\phi^\dagger$  transform as holomorphic adjoint fields with respect to the  $V$ -transformation

$$\phi^\dagger \rightarrow V\phi^\dagger V^{-1}, \quad \bar{\partial}\phi^\dagger \rightarrow V\bar{\partial}\phi^\dagger V^{-1}. \quad (5.4)$$

then the above equation can be further rewritten as

$$(\mathcal{D}_v\bar{\partial}\phi^\dagger)^\dagger = \frac{e^2}{2}\langle\Omega_s^{-1}\Omega_0\phi\rangle_G, \quad (5.5)$$

where we have used the holomorphic covariant derivative (4.10) and

$$\Omega_s^{-1}\langle X\Omega_s^{-1}\rangle_G\Omega_s = \text{Tr}[X\Omega_s^{-1}t^\alpha]\Omega_s^{-1}t^\alpha\Omega_s = \text{Tr}[\Omega_s^{-1}X\hat{t}^\alpha]\hat{t}^\alpha = \langle\Omega_s^{-1}X\rangle_G, \quad (5.6)$$

with  $\hat{t}^\alpha \equiv \Omega_s^{-1}t^\alpha\Omega_s$ .

We compare Eq. (5.5) with Eq. (3.33) by changing the variable from the adjoint field  $\chi_a$  to a gauge invariant field given by

$$\psi^\dagger \equiv S_s\chi_a S_s^{-1}. \quad (5.7)$$

With respect to  $\psi$ , Eq. (3.33) is expressed as

$$(\mathcal{D}_v\bar{\partial}\psi^\dagger)^\dagger = \frac{e^2}{2}\text{Tr}[\Omega^{-1}\Omega_0\psi\tilde{t}^\alpha]\tilde{t}^\alpha = \frac{e^2}{2}\langle\Omega^{-1}\Omega_0\psi\rangle_G, \quad (5.8)$$

with  $\tilde{t}^\alpha \equiv (S_s^\dagger)^{-1}t^\alpha S_s^\dagger$ . Thus, the fluctuation  $\phi$  of the master equation and the zero-mode  $\psi$  of the adjoint operator  $\mathbb{D}^\dagger$  obey exactly the same equations (5.5) and (5.8).

Since we have already proven that  $\psi = 0$  ( $\chi_a = 0$ ) in Sec. 3.2.2, we can immediately conclude that  $\phi = 0$ . Thus we have proven the uniqueness of the master equation. Then we conclude that all the zero-modes reside in the moduli matrix  $H_0$  and  $\Omega$  has no moduli parameters.

Note that this uniqueness is *local* uniqueness since  $\phi$  is only a small fluctuation around the true solutions. We still do not provide any proof for the global uniqueness. This is one of the important future problems. See however the discussion in Sec. 6.

## 5.2 Chern-Simons-Higgs theory

As in the YM case, we take equal couplings  $\kappa = \mu$  and consider a small fluctuation around the moduli matrix field  $\Omega \rightarrow \Omega_s + \delta\Omega$  which by plugging into the master equation (4.22) yields to linear order

$$(\mathcal{D}_v \bar{\partial} \phi^\dagger)^\dagger = \frac{16\pi^2}{\kappa^2} \Omega_s^{-1} \left( \langle \Omega_0 \Omega_s^{-1} \langle \Omega_0 \phi \Omega_s^{-1} \rangle_G \rangle_G + \langle \Omega_0 \phi \Omega_s^{-1} \langle \Omega_0 \Omega_s^{-1} - v^2 \mathbf{1}_N \rangle_G \rangle_G \right) \Omega_s ,$$

where we have used the identity (2.45) before inserting the fluctuations and we have again used the gauge invariant field  $\phi \equiv \Omega_s^{-1} \delta\Omega$ . As in the YM case we can use (5.6) to rewrite the above equation as

$$(\mathcal{D}_v \bar{\partial} \phi^\dagger)^\dagger = \frac{16\pi^2}{\kappa^2} \left( \langle \Omega_s^{-1} \Omega_0 \langle \Omega_s^{-1} \Omega_0 \phi \rangle_G \rangle_G + \langle \Omega_s^{-1} \Omega_0 \phi \langle \Omega_s^{-1} \Omega_0 - v^2 \mathbf{1}_N \rangle_G \rangle_G \right) . \quad (5.9)$$

Next, we would like to compare Eq. (5.9) with Eq. (3.50) which the zero-modes  $\chi_a$  of  $\mathbb{D}^\dagger$  in Eq. (3.46) obey, as in the case of the YM system. Using again the change of variables (5.7), we can write Eq. (3.50) on the form

$$(\mathcal{D}_v \bar{\partial} \psi^\dagger)^\dagger = \frac{16\pi^2}{\kappa^2} \left( \langle \Omega_s^{-1} \Omega_0 \langle \Omega_s^{-1} \Omega_0 \psi \rangle_G \rangle_G + \langle \Omega_s^{-1} \Omega_0 \psi \langle \Omega_s^{-1} \Omega_0 - v^2 \mathbf{1}_N \rangle_G \rangle_G \right) , \quad (5.10)$$

where  $\langle X \rangle_G = \text{Tr} [X \tilde{t}^\alpha] \tilde{t}^\alpha$  with  $\tilde{t}^\alpha \equiv (S^\dagger)^{-1} t^\alpha S^\dagger$ . Hence, the fluctuation  $\phi$  of the master equation and the zero-mode  $\psi$  of the adjoint operator  $\mathbb{D}^\dagger$  obey exactly the same equations, namely Eqs. (5.9) and (5.10).

Although  $\psi$  and  $\phi$  obey exactly the same equation, they are different fields. While  $\phi$  is a small fluctuation around  $\Omega_s = S_s S_s^\dagger$  which is related to the physical field configurations  $H$  and  $A_i$  through Eq. (4.1),  $\psi$  contains the normalizable zero-modes of  $\mathbb{D}^\dagger$  and has nothing to do with  $H$  and  $A_i$ . This difference must be carefully dealt with, because  $\phi$  is not necessarily a bounded solution especially in the case of the non-topological solitons. What we should require the normalizability of, is not the small fluctuations of  $\Omega$  but of the physical fields  $A_i$  and  $H$ , namely  $\delta A_i$  and  $\delta H$  must be normalizable.

To be concrete, let us first try to find additional moduli parameters of non-topological solitons in the Abelian CS case [11] by taking advantage of the moduli matrix method<sup>12</sup>. Given a moduli matrix  $H_0(z) = 1$ , we consider a fluctuation  $\phi = \omega^{-1} \delta\omega$  which asymptotically satisfies

$$\partial \bar{\partial} \phi = \frac{4\pi^2}{\kappa^2} (2\omega^{-1} - v^2) \omega^{-1} \phi \simeq -\frac{4\pi^2 v^2}{\kappa^2} \frac{C}{|z|^{2\alpha}} \phi , \quad (5.11)$$

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<sup>12</sup>Here we do not fix any gauge unlike Ref. [11] where the authors worked in the Coulomb gauge. Instead, we deal with the gauge invariant quantity, namely  $\omega$  and its fluctuation.

where we have used  $\omega \simeq |z|^{2\alpha}/C$ . Furthermore, the variation in the magnetic flux density is given by

$$F_{12} + \delta F_{12} = -2\bar{\partial}\partial \log \omega - 2\bar{\partial}\partial \phi \rightarrow -2\bar{\partial}\partial \log \frac{|z|^{2\alpha}}{C} - 2\bar{\partial}\partial \phi, \quad \text{for } |z| \rightarrow \infty. \quad (5.12)$$

In order not to change the total energy of the soliton, the second term must vanish. Therefore, the fluctuation  $\phi$  will asymptotically approach the real part of the holomorphic function as

$$\phi \rightarrow F_0(z) + F_0^*(\bar{z}), \quad F_0(z) = \sum_{i=1}^f a_i z^i. \quad (5.13)$$

From Eq. (5.11),  $f \in \mathbb{N}_{\geq 1}$  should be  $f < 2\alpha$ . Note that we have suppressed the constant term ( $i = 0$ ). This is because the asymptotic behavior of  $\omega$  is fixed as  $\omega \rightarrow |z|^{2\alpha}/C$ . For instance, if we consider  $F_0 = a_0$  ( $a_0 \in \mathbb{C}$ ), we have  $\omega' = \omega + \delta\omega \rightarrow \omega(1 + 2\text{Re}[a_0]) \simeq \frac{1+2\text{Re}[a_0]}{C} |z|^{2\alpha}$ . Since we have chosen, however, the asymptotic behavior  $\omega \rightarrow |z|^{2\alpha}/C$  in such a way that the configuration becomes regular, we have to choose  $a_0 = 0$ . Finally, we should impose the normalizability condition on  $\delta H$  and  $\delta \bar{A}$ . Fluctuations of  $H$  and  $\bar{A}$  are given by

$$\delta H = -FH, \quad \delta \bar{A} = -i\bar{\partial}F, \quad F \equiv s^{-1}\delta s. \quad (5.14)$$

$\phi$  and  $F$  are related by  $\phi = F + F^*$ . Thus we can asymptotically identify  $F \rightarrow F_0(z)$  and in turn  $\delta \bar{A} \rightarrow 0$  is automatically ensured. Moreover, normalizability of  $\delta H$  which behaves as  $\delta H \rightarrow -F_0|z|^{-\alpha}$  requires that the power of  $F_0$  should be  $f = \hat{\alpha} - 1$ . Hence, we obtain the boundary condition of  $\phi$  and  $\omega$

$$\phi \rightarrow \text{Re} \left[ \sum_{i=1}^{\hat{\alpha}-1} a_i z^i \right], \quad (5.15)$$

$$\omega \rightarrow C^{-1}|z|^{2\alpha} \left( 1 + \sum_{i=1}^{\hat{\alpha}-1} a_i z^i + \text{c.c.} \right). \quad (5.16)$$

This supplies  $\hat{\alpha} - 1$  additional complex parameters  $\{a_i\}$  ( $i = 1, 2, \dots, \hat{\alpha} - 1$ ) in  $\omega$  as varieties of the boundary condition for  $\omega$ . Note that this result is not consistent with the index theorem result  $\mathcal{I} = \hat{\alpha}$  ( $N_f = N = n_0 = 1$ ) given in Eq. (3.44). This mismatch was first observed in Ref. [9]. In this way, the index sometimes over-counts the zero-modes. So we should be careful when we count the physical zero-modes using the index theorem, i.e. we should correctly subtract unphysical zero-modes. This problem seems however only to arise in the non-topological cases.

Therefore, we conclude that  $\omega$  has its own zero-modes in addition to those in the moduli matrix for the non-topological solitons.

This is in sharp contrast to the case of the topological CS solitons in which there are no degrees of freedom in  $\omega$  once the moduli matrix is given. This can be seen as follows. The fluctuation  $\phi$  asymptotically satisfies

$$\bar{\partial}\partial\phi \simeq m_{\kappa\infty}^2\phi, \quad (5.17)$$

where we have used the asymptotic form of  $\omega \sim |z|^{2k}/v^2$ . Clearly, the right-hand side does not allow  $\phi$  to be a harmonic function and there is no normalizable solution compatible with the fact that  $\phi$  needs to be the real part of a holomorphic polynomial and hence we can conclude that  $\phi$  must vanish.

Once we fix the boundary condition of  $\omega$ , namely we set  $\phi \rightarrow 0$ , we can utilize the similarity between  $\phi$  and  $\psi$  which both satisfy the boundary condition  $\phi, \psi \rightarrow 0$  at infinity. Since we have already shown that plausibly  $\psi = 0$  ( $\chi_a = 0$ ) in Sec. 3.3.2, we expect in general that  $\phi = 0$ , so that the master equation is unique. Note that this uniqueness is local uniqueness since  $\phi$  is only a small fluctuation around the true solutions. As in the case of YM solitons, the global uniqueness problem is an important future problem.

Let us next consider the non-topological non-Abelian CS solitons. As in the Abelian case, we consider a variation in the magnetic flux density

$$\delta\text{Tr}[F_{12}] = -2\delta(\partial\bar{\partial}\text{Tr}\log\Omega) = -2\partial\bar{\partial}\text{Tr}[\phi], \quad (5.18)$$

where  $\phi = \Omega_s^{-1}\delta\Omega$  is an  $N$ -by- $N$  matrix field. The contribution to the total magnetic flux from  $\phi$  must be zero, so we require the following boundary condition for  $\phi$ :

$$\phi \rightarrow \Omega_s^{-1}F_0(z)\Omega_s + F_0^\dagger(\bar{z}), \quad (5.19)$$

where  $F_0(z)$  is an arbitrary  $N$ -by- $N$  holomorphic matrix. With this boundary condition, we have to solve Eq. (5.9). This holomorphic matrix  $F_0(z)$  supplies additional zero-modes to the non-topological solitons. In order to find a condition on  $F_0(z)$ , let us consider the fluctuation of  $\bar{A}$  and  $H$  which can be written as

$$\delta H = -S^{-1}FSH, \quad \delta\bar{A} = -iS^{-1}\bar{\partial}FS, \quad F \equiv \delta SS^{-1}. \quad (5.20)$$

Note that  $F$  is related to  $\phi$  by

$$\phi = \Omega_s^{-1}F\Omega_s + F^\dagger. \quad (5.21)$$

Comparing this with Eq. (5.19), we can identify  $F_0(z)$  as the asymptotic function of  $F$ , namely  $F(z, \bar{z}) \rightarrow F_0(z)$  as  $|z| \rightarrow \infty$ . The fluctuations of the physical fields  $H$  and  $\bar{A}$  must asymptotically go to zero. The holomorphy of  $F_0(z)$  is indeed needed for  $\delta\bar{A}$  to vanish asymptotically.

Furthermore, powers of the holomorphic functions in  $F_0(z)$  are determined by the normalizability condition on  $\delta H$ .

Let us give some examples in order to understand better the situation. For simplicity, let us take  $G = U(2)$  and consider the non-topological soliton in the unbroken vacuum  $H = \text{diag}(0, 0)$ . To create the vortex, we take the moduli matrix and make the diagonal Ansatz for  $\Omega_s$

$$H_0 = \text{diag}(1, 0) , \quad \Omega_s = \text{diag}(\omega_1, 1) . \quad (5.22)$$

We choose the asymptotic behavior  $\omega_1 \sim |z|^{2\alpha}/C$  as  $|z| \rightarrow \infty$ . Note that this moduli matrix does not completely fix the  $V$ -equivalence. The residual infinitesimal  $V$ -transformation takes the form

$$V = \begin{pmatrix} 1 & \delta_1(z) \\ 0 & 1 + \delta_2(z) \end{pmatrix} . \quad (5.23)$$

This transforms  $\Omega_s$  as

$$\delta\Omega_s = \begin{pmatrix} 0 & \delta_1 \\ \delta_1^* & \delta_2 + \delta_2^* \end{pmatrix} . \quad (5.24)$$

Now we are ready to look at the fluctuations at infinity. To this end, we write  $F_0(z)$  as

$$F_0 = \begin{pmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z) \end{pmatrix} . \quad (5.25)$$

Then the fluctuations are expressed by

$$\delta\Omega = F\Omega_s + \Omega_s F^\dagger \rightarrow \begin{pmatrix} \omega_1(f_{11} + f_{11}^*) & \omega_1 f_{21}^* + f_{12} \\ \omega_1 f_{21} + f_{12}^* & f_{22} + f_{22}^* \end{pmatrix} , \quad (5.26)$$

$$\delta H = -S^{-1} F H_0 \rightarrow - \begin{pmatrix} \omega_1^{-1/2} f_{11} & 0 \\ f_{21} & 0 \end{pmatrix} . \quad (5.27)$$

We impose the square integrability condition on the fluctuations

$$\text{Tr} [\delta H \delta H^\dagger] \rightarrow \text{Tr} [\Omega_s^{-1} F_0 \Omega_0^\dagger F_0^\dagger] = \omega_1^{-1} |f_{11}|^2 + |f_{21}|^2 . \quad (5.28)$$

Hence, we find that the most generic form of the fluctuations is given as

$$f_{11} = \sum_{i=1}^{\hat{\alpha}-1} a_i z^i , \quad f_{21} = 0 . \quad (5.29)$$

On the other hand, there are no constraints for  $f_{12}$  and  $f_{22}$ , it seems that  $f_{12}$  and  $f_{22}$  have an infinite number of additional zero-modes. However, they are unphysical because they can be

eliminated by the residual  $V$ -transformation given in Eq. (5.24). Hence, we have found  $\hat{\alpha} - 1$  additional moduli parameters in  $f_{11}(z)$ . Furthermore, we have found the orientational moduli parameter  $b$  given in Eq. (4.35). So in total we have found  $\hat{\alpha} - 1 + 1 = \hat{\alpha}$  complex parameters for the non-topological solitons. Again, the result is not consistent with the index theorem result  $\mathcal{I} = 2\hat{\alpha}$  given in Eq. (3.44). At this stage, we are not certain if the index over-counts the number of physical zero-modes  $\hat{\alpha}$  by including unphysical modes or we simply did not exhaust all the possible zero-modes in the above moduli matrix calculation. We will leave this problem for a future work.

Let us consider another configuration by taking the following moduli matrix

$$H_0 = \text{diag}(1, 1) , \quad \Omega_s = \text{diag}(\omega_1, \omega_2) . \quad (5.30)$$

We fix the boundary conditions as  $\omega_1 \rightarrow |z|^{2\alpha_1}/C_1$  and  $\omega_2 \rightarrow |z|^{2\alpha_2}/C_2$ , while we define  $\alpha \equiv \alpha_1 + \alpha_2$ . Note that there are no residual  $V$ -equivalence relations in this case. The fluctuations are written as (where  $F_0(z)$  still is given by Eq. (5.25))

$$\delta\Omega = F\Omega_s + \Omega_s F^\dagger \rightarrow \begin{pmatrix} \omega_1(f_{11} + f_{11}^*) & \omega_1 f_{21}^* + \omega_2 f_{12} \\ \omega_1 f_{21} + \omega_2 f_{12}^* & \omega_2(f_{22} + f_{22}^*) \end{pmatrix} , \quad (5.31)$$

$$\delta H = -S^{-1} F H_0 \rightarrow - \begin{pmatrix} \omega_1^{-1/2} f_{11} & \omega_1^{-1/2} f_{12} \\ \omega_2^{-1/2} f_{21} & \omega_2^{-1/2} f_{22} \end{pmatrix} . \quad (5.32)$$

As before, we impose the square integrability condition

$$\text{Tr} [\delta H \delta H^\dagger] \rightarrow \text{Tr} [\Omega_s^{-1} F_0 \Omega_0^\dagger F_0^\dagger] = \omega_1^{-1} (|f_{11}|^2 + |f_{12}|^2) + \omega_2^{-1} (|f_{21}|^2 + |f_{22}|^2) . \quad (5.33)$$

This gives the upper bounds for  $f_{IJ}$  ( $I, J = 1, 2$ )

$$f_{II} = \sum_{i=1}^{\hat{\alpha}_I-1} a_i^{(II)} z^i , \quad (5.34)$$

$$f_{IJ} = \sum_{i=0}^{\hat{\alpha}_I-1} a_i^{(IJ)} z^i , \quad (I \neq J) . \quad (5.35)$$

Here we did not count  $i = 0$  for  $f_{II}$  because the constant part is fixed by the boundary condition  $\omega_I \rightarrow |z|^{2\alpha_I}/C_I$ . Thus we found  $(\hat{\alpha}_1 - 1) + \hat{\alpha}_1 + (\hat{\alpha}_2 - 1) + \hat{\alpha}_2 = 2(\hat{\alpha}_1 + \hat{\alpha}_2 - 1)$  moduli parameters in  $\{f_{IJ}\}$ . Again, there is a mismatch with the index theorem result (3.44), see the comments above.

## 6 Conclusion and discussion

In this paper we have studied the non-Abelian solitons in the  $\mathcal{N} = 2$  supersymmetric gauge theories in three dimensions; viz. the YM and CS gauge theories. We have used a common framework for investigation of the non-Abelian solitons which nicely works both for the YM theories and CS theories. We have found several new solitons; the non-Abelian non-topological solitons and the non-Abelian semi-local (topological/non-topological) solitons in the unbroken and the partially broken vacua which exist in the CS theories. The non-Abelian non-topological solitons have internal orientational moduli analogous to the non-Abelian topological solitons, which however is in general different from that of the topological solitons. Furthermore, to the best of our knowledge, the present paper is the first work which has found the semi-local solitons in the models with the number of flavors less than or equal to the number of colors.

In addition to the minimal solution, we have also paid great attention to the multiple solitons and their zero-modes. To this end, we have made use of two methods, counting the number of zero-modes by the index theorem and realizing the explicit zero-modes by the moduli matrix formalism. Those for the YM solitons given in Sec. 3.2 and 4.1 are all known and we have simply reviewed them. What we learned from these review parts of the YM vortices is that the moduli matrix formalism is a rigorous tool for investigating topological solitons<sup>13</sup>. Indeed, the index theorem counting [17] and the number of the zero-modes realized in the moduli matrix formalism [30] do in fact coincide. In order to confirm that the moduli matrix formalism indeed provides us with all possible moduli parameters, we have to solve the long-standing uniqueness and existence problems of the master equations (4.15) and (4.16). As to the uniqueness problem, we have made a progress. We have proven the local uniqueness of the master equation by finding a one-to-one correspondence between the small fluctuations of the master equations and the vanishing theorem.

We have also tried to count the zero-modes by the index theorem for the CS theories, although we could not prove the vanishing theorem. Furthermore, in order to go beyond the parameter counting, we have also applied the moduli matrix formalism which has already been used to investigate the topological solitons in Refs. [26, 27]. For the topological solitons, we have reached at the same level as the YM case, namely the coincidence of the index theorem result and the number of the zero-modes realized in the moduli matrix. Actually, we have found no differences both in the index theorem calculation and the moduli matrix formalism for the topological solitons

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<sup>13</sup>Needless to say, the moduli matrix formalism has been already established. Many results obtained using the moduli matrix formalism are summarized in the review [20].

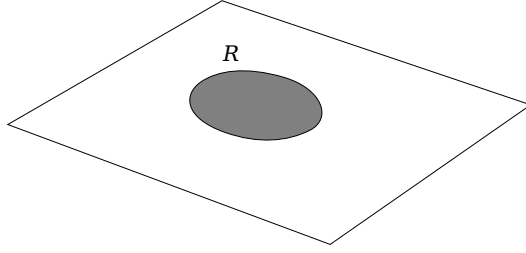


Fig. 1: The fluctuations of the master equation outside the radius  $R$  have to be infinitesimal, since the fluctuations have to vanish identically at infinity. Inside  $R$  they could become finite. However, we have shown in Sec. 5 that the infinitesimal variation has to vanish.

between the YM and CS theories, except for some technical details. This fact strongly suggests that the moduli matrix formalism is also a rigorous tool for investigating the CS solitons. If this is the case, the moduli space of the topological CS solitons is completely the same as that of the YM theory which has been studied intensively, which was conjectured in [26]. Even the metric of the moduli spaces are coincident in the two theories (the dynamics is however different due to the CS term [24]), so there is nothing new about the moduli space of the topological solitons.

However, this is not the end of story. Proper features of the CS theory appear when we study the non-topological and semi-local solitons in the unbroken and the partially broken vacua. We have also made use of the moduli matrix formalism and showed it works well too. A remarkable contrast to the YM theory is that the gauge invariant field  $\Omega$  has its own moduli parameters in addition those provided by the moduli matrix, namely, there are zero-modes in the master equation. We have found them in the varieties of the harmonic boundary condition for the master equation. Once  $H_0$  and the boundary condition of  $\Omega$  are properly fixed, we would expect in general that the master equation in the CS theory is unique at least locally. We have given several examples,  $G = U(2), U(4), U(1) \times SO(4)$  and  $U(1) \times USp(4)$  where the Abelian solitons can be embedded solutions in the special case of equal CS couplings. The generalization to the higher rank gauge groups is straightforward.

We have attacked the long-standing problem of the uniqueness properties of the master equations and we have proved completely independent of the gauge group that a small fluctuation of the gauge invariant field  $\Omega$  around the topological solitons is in one-to-one correspondence with the vanishing theorem of the index theorem calculation.

There is one subtlety, but as we shall argue, physically not too worrying fact about this uniqueness calculation. In this discussion here, we will consider only the question of local uniqueness proved for the YM case and expected for the topological CS case. The reason for which we



only claim that the uniqueness is local is that the fluctuations of the master equation considered are just infinitesimal fluctuations. Hence, one would immediately ask, what if the fluctuation is finite? Is there then a possibility for the existence of another solution and hence a parameter governing the different solutions? Let us first recall that in the vacuum, the solution to the master equation is completely unique in all cases, i.e. it is the well-known lump solution (even if it is technically singular in the local vortex case). Hence at an infinite radius, the fluctuations must vanish identically. Now let us assume that there exist two solutions,  $\Omega_{A,B}$ . The finite difference is necessarily zero at infinity. Consider now a finite radius  $R$  (see Fig. 1) much larger than the typical scale of the vortex system  $\mu$ . Let us assume that the finite difference could be non-zero within the radius  $R$ . However, from that radius till infinity the difference is expected to be very small, say of the order of  $1/(\mu R) \ll 1$ . Thus we expect the difference to be exactly an infinitesimal deviation from the true solution. However, our uniqueness calculation shows that if the vanishing theorem is satisfied, then there cannot be a non-zero fluctuation deviating from a true solution. The somewhat physical argument tells us that in this case, we can expect the difference  $\Omega_B - \Omega_A$  to vanish. A more rigid demonstration of this difficult problem is indeed welcome and left for future works.

We will now discuss what we were not able to do in this work. For the YM theory, the global uniqueness and the existence problems of the master equation are unsolved<sup>14</sup>. For the CS theory, 1) the proof of the vanishing theorem, 2) the uniqueness and existence problems, 3) matching between the parameter counting of the index theorem and the moduli matrix formalism for the non-topological and semi-local solitons, 4) generic solutions of the non-topological and semi-local solitons, especially with the intermediate relative orientational moduli and 5) numerical solutions for the non-topological and semi-local vortices for generic coupling  $\kappa \neq \mu$ . We left these problems for future works.

A comment in store is about the equal gauge coupling choice which we have made in this paper. It causes no problem for the number of moduli in a solution. Our derivations of all the generic formulae done for equal gauge couplings can trivially be extended to the generic gauge coupling case. We have made this choice for simplicity and aesthetic beauty. However, the solution to a given master equation (CS or YM case) is clearly different in the case of different gauge couplings. Hence, for the realization of explicit solutions, it is certainly important to consider the generic gauge coupling case and verify the existence of the solutions for all (finite) values of the couplings and values of the moduli parameters.

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<sup>14</sup>Clearly, the existence of the embedded Abelian solution exists provided the gauge couplings are equal. What we mean here is that the solution exists in general for all values of the couplings and moduli parameters.

Another interesting thing might be the realization of the low-energy effective theory on the non-topological CS solitons. For the topological CS solitons it has been studied in Refs. [21, 24, 25]. Dynamics and interactions of non-Abelian solitons, especially non-topological solitons, which may depend on the orientations are also interesting open problems.

Non-Abelian solitons in the CS theory with the YM kinetic term may have similar properties compared with the solitons in the current work. The D-brane construction of the moduli space is another interesting problem. Furthermore, the non-relativistic limit may be also interesting, especially due to the integrability found in the Abelian non-relativistic systems [57].

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